Mathematics 2017 (Outside Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. **If for any 2×2 square matrix A,**

$A(\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$,

then write the value of $|A|$. [1]

Solution : We have,

$$A(\text{adj } A) = \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}$$

As, $A(\text{adj } A) = |A|\, I$

$$\begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = 8\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

On comparing, we get

$$|A| = 8.$$ **Ans.**

2. **Determine the value of 'k' for which the following function is continuous at $x = 3$.**

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$ [1]

Solution : Given,

$$f(x) = \begin{cases} \dfrac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$$

Since $f(x)$ is continuous at $x = 3$

$\therefore \quad \lim_{x \to 3} f(x) = f(3)$

$\Rightarrow \quad \lim_{x \to 3} f(x) = k$

$\Rightarrow \quad \lim_{x \to 3} \dfrac{(x+3)^2 - 36}{x-3} = k$

$\Rightarrow \quad \lim_{x+3 \to 6} \dfrac{(x+3)^2 - 6^2}{(x+3)-6} = k$

$\Rightarrow \quad \lim_{x \to 3} x + 3 + 6 = k$

$\Rightarrow \quad 12 = k$

Thus, $f(x)$ is continuous at $x = 3$; if $k = 12$. **Ans.**

3. **Find : $\int \dfrac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$.** [1]

Solution : We have,

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx = -2\int \frac{\cos^2 x - \sin^2 x}{2 \sin x \cos x} dx$$

$$= -2\int \frac{\cos 2x}{\sin 2x} dx$$

$$= -2\int \cot 2x \, dx$$

$$= -\log|\sin 2x| + C$$

Ans.

4. **Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$.** [1]

Solution : Since, the direction ratios of the normal to the given planes are proportional.

i.e., $\frac{2}{5} = \frac{-1}{-2.5} = \frac{2}{5}$

Thus, the given planes are parallel.

Now, let $P(x_1, y_1, z_1)$ be any point on $2x - y + 2z - 5 = 0$

Then, $2x_1 - y_1 + 2z_1 - 5 = 0$

or $5x_1 - 2.5y_1 + 5z_1 - 12.5 = 0$ ...(i)

The length of the perpendicular from $P(x_1, y_1, z_1)$ to $5x - 2.5y + 5z - 20 = 0$,

$$= \left|\frac{5x_1 - 2.5y_1 + 5z_1 - 20}{\sqrt{(5)^2 + (-2.5)^2 + (5)^2}}\right|$$

$$= \left|\frac{12.5 - 20}{\sqrt{25 + 6.25 + 25}}\right| \quad \text{[From eq. (i)]}$$

$$= \left|\frac{-7.5}{7.5}\right| = 1 \text{ unit}$$

Therefore, the distance between the given planes is 1 unit. **Ans.**

SECTION — B

5. If A is a skew-symmetric matrix of order 3, then prove that det A = 0. **[2]**

Solution : Given, A is a skew-symmetric matrix of order 3.

So, $A^T = -A$

Now, $|A^T| = |-A|$

$|A^T| = (-1)^3 |A|$

$[\because |kA| = k^n |A|$ where n is order of A$]$

$|A| = -|A| \quad [\because |A^T| = |A|]$

$\Rightarrow |A| + |A| = 0$

$\therefore 2|A| = 0$ or $|A| = 0$.

i.e., det A = 0 **Hence Proved.**

6. Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $[-\sqrt{3}, 0]$ **[2]**

Solution : We know that the polynomial function $f(x) = x^3 - 3x$ is everywhere continuous and differentiable.

So, $f(x)$ is continuous on $[-\sqrt{3}, 0]$.

Also, $f(x)$ is differentiable on $(-\sqrt{3}, 0)$

Now, $f(-\sqrt{3}) = (-\sqrt{3})^3 - 3(-\sqrt{3})$

$= -3\sqrt{3} + 3\sqrt{3} = 0$

and $f(0) = (0)^3 - 3(0) = 0$

$\therefore f(-\sqrt{3}) = f(0)$

Thus, all the three conditions of Rolle's theorem are satisfied.

Now, there must exist $c \in (-\sqrt{3}, 0)$ such that $f'(c) = 0$

Now, $f'(x) = 3x^2 - 3$

Then, $f'(c) = 0$

$\Rightarrow 3c^2 - 3 = 0$

$\Rightarrow 3(c^2 - 1) = 0$

$\Rightarrow c = \pm 1$

$c \neq 1$ as $1 \notin (-\sqrt{3}, 0)$

$\therefore c = -1 \in (-\sqrt{3}, 0)$

Thus, required value of c is -1. **Ans.**

7. The volume of a cube is increasing at the rate of 9 cm³/s. How fast is its surface area increasing when the length of an edge is 10 cm ? **[2]**

Solution : Let, the side of the cube be a cm

then, volume of cube $(V) = a^3$

Differentiating V w.r.t. t, we get

$$\frac{dV}{dt} = 3a^2 \frac{da}{dt}$$

$\Rightarrow 3a^2 \frac{da}{dt} = 9 \text{ cm}^3/\text{s}. \quad \left[\text{Given, } \frac{dV}{dt} = 9 \text{ cm}^3/\text{s}\right]$

$\Rightarrow \frac{da}{dt} = \frac{3}{a^2} \text{ cm/s}$

and surface area of cube $(S) = 6a^2$

$$\frac{dS}{dt} = 12a \frac{da}{dt}$$

$$\frac{dS}{dt} = 12a \times \frac{3}{a^2} = \frac{36}{a} \text{ cm}^2/\text{s}$$

When $a = 10$

$\therefore \left[\frac{dS}{dt}\right]_{a=10} = \frac{36}{10} = 3.6 \text{ cm}^2/\text{s}.$ **Ans.**

8. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R. **[2]**

Solution : We have,

$f(x) = x^3 - 3x^2 + 6x - 100$

then, $f'(x) = 3x^2 - 6x + 6$

$= 3(x^2 - 2x + 1) + 3$

$= 3(x - 1)^2 + 3 > 0$ for all $x \in R$.

Hence, the function $f(x)$ is increasing on R.

Hence Proved.

9. The x-coordinate of a point on the line joining the points P(2, 2, 1) and Q (5, 1, – 2) is 4. Find its z-coordinate. [2]

Solution : Given, the points P(2, 2, 1) and Q (5, 1, – 2) of a line.

Then, equation of line PQ,

$$\frac{x-2}{5-2} = \frac{y-2}{1-2} = \frac{z-1}{-2-1}$$

$$\Rightarrow \quad \frac{x-2}{3} = \frac{y-2}{-1} = \frac{z-1}{-3} = \lambda$$

∴ Coordinates of any point of line PQ is

$$(3\lambda+2, -\lambda+2, -3\lambda+1)$$

Now, we have the x-coordinate as 4.

$$\Rightarrow \quad 3\lambda + 2 = 4$$

$$\Rightarrow \quad 3\lambda = 2$$

$$\therefore \quad \lambda = \frac{2}{3}$$

∴ z coordinate is $-3\lambda + 1$ *i.e.*, -1. **Ans.**

10. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events. [2]

Solution : Since, A be the event of number obtained is even

then, $A = \{2, 4, 6\}$

and B be the event of number obtained is red

then, $B = \{1, 2, 3\}$

$\therefore \quad A \cap B = \{2\}$

So, $P(A) = \frac{3}{6} = \frac{1}{2}$; $P(B) = \frac{3}{6} = \frac{1}{2}$; $P(A \cap B) = \frac{1}{6}$

Now, $P(A \cap B) \neq P(A) \cdot P(B)$

$$\frac{1}{6} \neq \frac{1}{4}$$

Hence, the events A and B are not independent events. **Ans.**

11. Two tailors, A and B earn ₹ 300 and ₹ 400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP. [2]

Solution : Let x and y be the number of days for which the tailors A and B work respectively.

Total cost per day = ₹ $(300x + 400y)$

Let Z denote the total cost in rupees, then,

$$Z = 300x + 400y$$

Since in one day 6 shirts are stitched by tailor A and 10 shirts are stitched by tailor B and it is desired to produce atleast 60 shirts.

$$\therefore \quad 6x + 10y \geq 60$$

It is given that 4 pairs of trousers are stitched by each tailor A and B per day to produce atleast 32 pairs of trousers.

$$\therefore \quad 4x + 4y \geq 32$$

Finally, the no. of shirts and pair of trousers cannot be negative.

$$\therefore \quad x \geq 0, \; y \geq 0$$

Thus, mathematical formulation of the given LPP is as follows :

Minimize $Z = 300x + 400y$

Subject to constraints :

$$6x + 10y \geq 60$$

$$4x + 4y \geq 32$$

$$x \geq 0, \; y \geq 0$$ **Ans.**

12. Find : $\int \frac{dx}{5 - 8x - x^2}$. [2]

Solution : We have,

$$\int \frac{dx}{5-8x-x^2} = \int \frac{dx}{5-8x-x^2+(4)^2-(4)^2}$$

$$= \int \frac{dx}{21-[(4)^2+8x+(x)^2]}$$

$$= \int \frac{dx}{(\sqrt{21})^2-(x+4)^2}$$

$$= \frac{1}{2\sqrt{21}} \log \left| \frac{\sqrt{21}+x+4}{\sqrt{21}-x-4} \right| + C$$ **Ans.**

SECTION — C

13. If $\tan^{-1}\frac{x-3}{x-4} + \tan^{-1}\frac{x+3}{x+4} = \frac{\pi}{4}$, then find the value of x. [4]

Solution : We have,

$$\tan^{-1}\left(\frac{x-3}{x-4}\right) + \tan^{-1}\left(\frac{x+3}{x+4}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x-3}{x-4}\right) + \tan^{-1}\left(\frac{x+3}{x+4}\right) = \tan^{-1} 1$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x-3}{x-4}\right) = \tan^{-1}1-\tan^{-1}\left(\frac{x+3}{x+4}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x-3}{x-4}\right) = \tan^{-1}\left(\frac{1-\frac{x+3}{x+4}}{1+1\times\frac{x+3}{x+4}}\right)$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{x-3}{x-4}\right) = \tan^{-1}\left(\frac{x+4-x-3}{x+4+x+3}\right)$$

$$\Rightarrow \quad \frac{x-3}{x-4} = \frac{1}{2x+7}$$

$$\Rightarrow \quad (2x+7)(x-3) = x-4$$

$$\Rightarrow \quad 2x^2-6x+7x-21 = x-4$$

$$\Rightarrow \quad 2x^2+x-21 = x-4$$

$$\Rightarrow \quad 2x^2 = 17$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{17}{2}}$$ **Ans.**

14. Using properties of determinants, prove that

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3.$$ **[4]**

Solution : Let $\Delta = \begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$ = L. H. S.

Applying, $R_1 \to R_1 - R_2$, $R_2 \to R_2 - R_3$

$$\Delta = \begin{vmatrix} a^2-2a & a-1 & 0 \\ 2a-1 & a-2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= (a-1)^2\begin{vmatrix} a+1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

[Taking $(a-1)$ common from R_1 and R_2]

Now, expanding along C_3,

$$\Delta = (a-1)^2\{(a+1)\cdot 1-(2)1\}$$

$$= (a-1)^2(a-1) = (a-1)^3$$

= R. H. S. **Hence Proved.**

OR

Find matrix A such that

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

Solution : We have,

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}_{3\times 2} A = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}_{3\times 2}$$

$\therefore$ A is of order 2×2

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then, $\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$

$$\begin{bmatrix} 2a-c & 2b-d \\ a & b \\ -3a+4c & -3b+4d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

By equality of matrices, on comparing, we get

$$2a-c = -1$$
$$2b-d = -8$$
$$a = 1$$
$$b = -2$$
$$-3a+4c = 9$$
$$-3b+4d = 22$$

On solving the equations, we get

$a = 1,\quad b = -2,\quad c = 3,\quad d = 4$

Hence, $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ **Ans.**

15. If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$. **[4]**

Solution : We have, $x^y + y^x = a^b$

$$\Rightarrow \quad e^{\log x^y} + e^{\log y^x} = a^b$$

$$\Rightarrow \quad e^{y\log x} + e^{x\log y} = a^b$$

On differentiating both sides w.r.t. x, we get

$$e^{y\log x}\left(y\cdot\frac{1}{x}+\log x\cdot\frac{dy}{dx}\right)+e^{x\log y}\left(x\cdot\frac{1}{y}\frac{dy}{dx}+\log y\right)=0$$

$$\Rightarrow \quad x^y\left(\frac{y}{x}+\log x\frac{dy}{dx}\right)+y^x\left(\frac{x}{y}\frac{dy}{dx}+\log y\right)=0$$

$$\Rightarrow \quad yx^{y-1}+x^y\log x\frac{dy}{dx}+xy^{x-1}\frac{dy}{dx}+y^x\log y=0$$

$$\Rightarrow \quad \frac{dy}{dx}(x^y\log x+xy^{x-1}) = -(y^x\log y+yx^{y-1})$$

$$\therefore \quad \frac{dy}{dx} = -\left(\frac{y^x\log y+y\cdot x^{y-1}}{x^y\log x+x\cdot y^{x-1}}\right)$$ **Ans.**

OR

If $e^y(x+1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Solution : We have,

$$e^y(x+1) = 1$$

$$\Rightarrow \quad e^y = \frac{1}{x+1}$$

$$\Rightarrow \quad \log e^y = \log_e\left(\frac{1}{x+1}\right)$$

[Taking log of both]

$$\Rightarrow \quad y = -\log(x+1)$$

On differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = -\frac{1}{x+1} \quad ...(i)$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \left(-\frac{1}{(x+1)}\right)^2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2 \quad \text{[using (i)]}$$

Hence Proved.

16. Find : $\int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}\,d\theta$ **[4]**

Solution :

$$\text{Let } I = \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}\,d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(5-4(-\sin^2\theta+1))}\,d\theta$$

$$= \int \frac{\cos\theta}{(4+\sin^2\theta)(1+4\sin^2\theta)}\,d\theta$$

Substituting $\sin\theta = y \Rightarrow \cos\theta\, d\theta = dy$

$$I = \int \frac{1}{(4+y^2)(1+4y^2)}\,dy$$

$$\text{Let } \frac{1}{(4+y^2)(1+4y^2)} = \frac{A}{(4+y^2)} + \frac{B}{(1+4y^2)}$$

$$1 = A(1+4y^2) + B(4+y^2)$$

Putting $y = 0$, we get

$$1 = A + 4B \quad ...(i)$$

Putting $y = 1$, we get

$$1 = 5A + 5B \quad ...(ii)$$

Solving (i) and (ii), we get

$$A = \frac{-1}{15} \text{ and } B = \frac{4}{15}$$

$$\therefore \quad I = \int\left(\frac{-1}{15(4+y^2)} + \frac{4}{15(1+4y^2)}\right)dy$$

$$\Rightarrow \quad I = \frac{-1}{15}\int\frac{dy}{((2)^2+y^2)} + \frac{4}{15}\int\frac{dy}{((1)^2+(2y)^2)}$$

$$\Rightarrow \quad I = \frac{-1}{15}\times\frac{1}{2}\tan^{-1}\frac{y}{2} + \frac{4}{15}\times\frac{1}{2}\tan^{-1}2y + C$$

$$\Rightarrow \quad I = \frac{-1}{30}\tan^{-1}\left(\frac{\sin\theta}{2}\right) + \frac{2}{15}\tan^{-1}(2\sin\theta) + C$$

Ans.

17. Evaluate : $\int_0^\pi \frac{x\tan x}{\sec x + \tan x}\,dx.$ **[4]**

Solution : Let $I = \int_0^\pi \frac{x\tan x}{\sec x+\tan x}\,dx. \quad ...(i)$

$$\text{Now,} \quad I = \int_0^\pi \frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}$$

$$\Rightarrow \quad I = \int_0^\pi \frac{(\pi-x)\tan x}{\sec x+\tan x}\,dx \quad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^\pi \frac{x\tan x}{\sec x+\tan x}\,dx + \int_0^\pi \frac{(\pi-x)\tan x}{\sec x+\tan x}\,dx$$

$$\Rightarrow \quad 2I = \pi\int_0^\pi \frac{\tan x}{\sec x+\tan x}\,dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^\pi \frac{\tan x}{\sec x+\tan x}\times\frac{\sec x-\tan x}{\sec x-\tan x}\,dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^\pi \frac{(\sec x\tan x-\tan^2 x)}{\sec^2 x-\tan^2 x}\,dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^\pi (\sec x\tan x - \sec^2 x + 1)\,dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\left[\sec x - \tan x + x\right]_0^\pi$$

$$\Rightarrow \quad I = \frac{\pi}{2}[(\sec\pi - \tan\pi + \pi) - (\sec 0 - \tan 0 + 0)]$$

$$\Rightarrow \quad I = \frac{\pi}{2}[(-1+\pi)-(1)]$$

$$\therefore \quad I = \frac{\pi(\pi-2)}{2}$$

Ans.

OR

Evaluate : $\int_1^4 \{|x-1|+|x-2|+|x-4|\}\,dx$

Solution : Let $I = \int_1^4 \{|x-1|+|x-2|+|x-4|\}\,dx$

$$\Rightarrow \quad I = \int_1^4 |x-1|\,dx + \int_1^4 |x-2|\,dx + \int_1^4 |x-4|\,dx$$

$$\Rightarrow \quad I = \int_1^4 |x-1|\,dx + \int_1^2 |x-2|\,dx + \int_2^4 |x-2|\,dx + \int_1^4 |x-4|\,dx$$

$\Rightarrow \quad I = \int_1^4 (x-1)\,dx - \int_1^2 (x-2)\,dx + \int_2^4 (x-2)dx - \int_1^4 (x-4)\,dx$

$\Rightarrow \quad I = \frac{1}{2}\left[(x-1)^2\right]_1^4 - \frac{1}{2}\left[(x-2)^2\right]_1^2 + \frac{1}{2}\left[(x-2)^2\right]_2^4 - \frac{1}{2}\left[(x-4)^2\right]_1^4$

$= \frac{1}{2}(9-0) - \frac{1}{2}(0-1) + \frac{1}{2}(4-0) - \frac{1}{2}(0-9)$

$\Rightarrow \quad I = \frac{9}{2}+\frac{1}{2}+\frac{4}{2}+\frac{9}{2}$

$\therefore \quad I = \frac{23}{2}$ **Ans.**

18. Solve the differential equation

$(\tan^{-1}x - y)\,dx = (1+x^2)\,dy.$ [4]

Solution : We have,

$$(\tan^{-1}x - y)\,dx = (1+x^2)\,dy$$

i.e., $\quad \frac{dy}{dx} = \frac{\tan^{-1}x - y}{1+x^2}$

$\Rightarrow \quad \frac{dy}{dx} + \left(\frac{1}{1+x^2}\right)y = \frac{\tan^{-1}x}{1+x^2}$

which is a linear differential equation of the form,

$$\frac{dy}{dx} + Py = Q$$

where, $P = \frac{1}{1+x^2}$ and $Q = \frac{\tan^{-1}x}{1+x^2}$

Now, $\text{I.F.} = e^{\int P dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$

Then, required solution is :

$$ye^{\tan^{-1}x} = \int e^{\tan^{-1}x}\frac{\tan^{-1}x}{1+x^2}dx + C$$

Putting, $\tan^{-1}x = t$

$\Rightarrow \quad \frac{1}{1+x^2}dx = dt$

$\Rightarrow \quad ye^t = \int e^t\, t\,dt + C$

$\Rightarrow \quad ye^t = t\int e^t dt - \int\left(\frac{d}{dt}(t)\int e^t\,dt\right)dt + C$

$\Rightarrow \quad ye^t = te^t - e^t + C$

$\Rightarrow \quad ye^t = (t-1)e^t + C$

$\Rightarrow \quad ye^{\tan^{-1}x} = (\tan^{-1}x - 1)\,e^{\tan^{-1}x} + C$

$[\because \; t = \tan^{-1}x]$

$\therefore \quad y = \tan^{-1}x - 1 + Ce^{-\tan^{-1}x}$

which is the required solution. **Ans.**

19. Show that the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence, find the area of the triangle. [4]

Solution : Given, the position vectors of the points A, B and C are $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$, respectively.

Then, $\vec{OA} = 2\hat{i}-\hat{j}+\hat{k}$,

$\vec{OB} = \hat{i}-3\hat{j}-5\hat{k}$

and $\vec{OC} = 3\hat{i}-4\hat{j}-4\hat{k}$

Now, $\vec{AB} = \vec{OB}-\vec{OA}$

$= (\hat{i}-3\hat{j}-5\hat{k})-(2\hat{i}-\hat{j}+\hat{k})$

$= -\hat{i}-2\hat{j}-6\hat{k}$

$\vec{BC} = \vec{OC}-\vec{OB}$

$= (3\hat{i}-4\hat{j}-4\hat{k})-(\hat{i}-3\hat{j}-5\hat{k})$

$= 2\hat{i}-\hat{j}+\hat{k}$

and $\vec{CA} = \vec{OA}-\vec{OC}$

$= (2\hat{i}-\hat{j}+\hat{k})-(3\hat{i}-4\hat{j}-4\hat{k})$

$= -\hat{i}+3\hat{j}+5\hat{k}$

$|\vec{AB}|^2 = (-1)^2+(-2)^2+(-6)^2 = 1+4+36 = 41$

$|\vec{BC}|^2 = (2)^2+(-1)^2+(1)^2 = 4+1+1 = 6$

and $|\vec{CA}|^2 = (-1)^2+(3)^2+(5)^2 = 1+9+25 = 35$

$\therefore \quad |\vec{AB}|^2 = |\vec{BC}|^2 + |\vec{CA}|^2$

Hence, A, B, C are the vertices of a right angled triangle.

Now, area of $\Delta ABC = \frac{1}{2}\left|\vec{BC}\times\vec{CA}\right|$

$\therefore \quad \vec{BC}\times\vec{CA} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ -1 & 3 & 5 \end{vmatrix}$

$= -8\hat{i}-11\hat{j}+5\hat{k}$

$$|\vec{BC} \times \vec{CA}| = \sqrt{(-8)^2 + (-11)^2 + (5)^2}$$

$$= \sqrt{64 + 121 + 25} = \sqrt{210}$$

Hence, Area of $\Delta ABC = \frac{\sqrt{210}}{2}$ square units. **Ans.**

20. Find the value of λ, if four points with position vectors $3\hat{i} + 6\hat{j} + 9\hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$, $2\hat{i} + 3\hat{j} + \hat{k}$ and $4\hat{i} + 6\hat{j} + \lambda\hat{k}$ are coplanar. [4]

Solution : Let, the four points be A, B, C and D with position vectors $3\hat{i} + 6\hat{j} + 9\hat{k}$, $\hat{i} + 2\hat{j} + 3\hat{k}$, $2\hat{i} + 3\hat{j} + \hat{k}$ and $4\hat{i} + 6\hat{j} + \lambda\hat{k}$, respectively.

Then, $\vec{OA} = 3\hat{i} + 6\hat{j} + 9\hat{k}$

$\vec{OB} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{OC} = 2\hat{i} + 3\hat{j} + \hat{k}$

and $\vec{OD} = 4\hat{i} + 6\hat{j} + \lambda\hat{k}$

Then, $\vec{AB} = \vec{OB} - \vec{OA}$

$$= (\hat{i} + 2\hat{j} + 3\hat{k}) - (3\hat{i} + 6\hat{j} + 9\hat{k})$$

$$= -2\hat{i} - 4\hat{j} - 6\hat{k}$$

$\vec{AC} = \vec{OC} - \vec{OA}$

$$= (2\hat{i} + 3\hat{j} + \hat{k}) - (3\hat{i} + 6\hat{j} + 9\hat{k})$$

$$= -\hat{i} - 3\hat{j} - 8\hat{k}$$

and $\vec{AD} = \vec{OD} - \vec{OA}$

$$= (4\hat{i} + 6\hat{j} + \lambda\hat{k}) - (3\hat{i} + 6\hat{j} + 9\hat{k})$$

$$= \hat{i} + (\lambda - 9)\hat{k}$$

Since, the points are coplanar, then

$$[\vec{AB}, \vec{AC}, \vec{AD}] = 0$$

i.e. $$\begin{bmatrix} -2 & -4 & -6 \\ -1 & -3 & -8 \\ 1 & 0 & \lambda - 9 \end{bmatrix} = 0$$

$\Rightarrow -2(-3\lambda + 27) + 4(-\lambda + 9 + 8) - 6(3) = 0$

$\Rightarrow 6\lambda - 54 - 4\lambda - 68 - 18 = 0$

$\Rightarrow 2\lambda - 4 = 0$

$\Rightarrow 2\lambda = 4$

$\therefore \lambda = 2$ **Ans.**

21. There are 4 cards numbered 1, 3, 5 and 7, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X. [4]

Solution : Given, X denote the sum of the numbers on the two drawn cards.

Then, X can take values 4, 6, 8, 10, 12

and sample space (S) = {(1,3), (1,5), (1,7), (3,1), (3,5), (3, 7), (5, 1), (5, 3), (5, 7), (7, 1), (7, 3), (7, 5)}

So, the probability distribution of X is as given below :

X	4	6	8	10	12
P(X)	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{4}{12}$	$\frac{2}{12}$	$\frac{2}{12}$

Computation of Mean and Variance :

x_i	$P(X = x_i) = p_i$	$p_i x_i$	$p_i x_i^2$
4	$\frac{2}{12}$	$\frac{8}{12}$	$\frac{32}{12}$
6	$\frac{2}{12}$	$\frac{12}{12}$	$\frac{72}{12}$
8	$\frac{4}{12}$	$\frac{32}{12}$	$\frac{256}{12}$
10	$\frac{2}{12}$	$\frac{20}{12}$	$\frac{200}{12}$
12	$\frac{2}{12}$	$\frac{24}{12}$	$\frac{288}{12}$
		$\Sigma p_i x_i = \frac{96}{12}$	$\Sigma p_i x_i^2 = \frac{848}{12}$

We have, $\Sigma p_i x_i = \frac{96}{12} = 8$

$\therefore$ Mean, $\bar{X} = \Sigma p_i x_i = 8$

Now, $Var(X) = \Sigma p_i x_i^2 - (\Sigma p_i x_i)^2$

$$= \frac{848}{12} - (8)^2$$

$$= \frac{848}{12} - 64$$

$$= \frac{80}{12} = \frac{20}{3}$$

Hence, Mean = 8 and Variance = $\frac{20}{3}$ **Ans.**

22. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that

70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance ? Is regularity required only in school ? Justify your answer. [4]

Solution : Consider the following events :

A : the student has grade A.

E_1 : the student has 100% attendance.

E_2 : the student is irregular.

Then, probability of the students having 100% attendance :

$$P(E_1) = 30\% = 0.3$$

Similarly, $P(E_2) = 70\% = 0.7$

Now, by previous year report, the probability of the students having grade A who have 100% attendance :

$$P(A/E_1) = 70\% = 0.7$$

and the probability of the students having grade A who are irregular :

$$P(A/E_2) = 10\% = 0.1$$

Then, the probability of the student having 100% attendance who already has attain A grade

$$= P(E_1/A)$$

By Bayes' theorem,

$$P(E_1/A) = \frac{P(A/E_1)P(E_1)}{P(A/E_1)P(E_1)+P(A/E_2)P(E_2)}$$

$$= \frac{0.7 \times 0.3}{0.7 \times 0.3 + 0.1 \times 0.7}$$

$$= \frac{21}{21+7}$$

$$= \frac{21}{28} = \frac{3}{4}$$

$$= 45\%$$

No, regularity is required in school as well as in life.

It helps to be disciplined in every aspect of life.

Or, when you work regularly, inspiration strikes regularly. **Ans.**

23. Maximise $Z = x + 2y$ subject to the constraints :

$$x + 2y \geq 100 \qquad 2x - y \leq 0$$

$$2x + y \leq 200 \qquad x, y \geq 0$$

Solve the above LPP graphically. [4]

Solution : Given,

Maximise $Z = x + 2y$

Subject to the constraints :

$$x + 2y \geq 100 \qquad 2x - y \leq 0$$

$$2x + y \leq 200 \qquad x, y \geq 0$$

Converting the inequations into equations we obtain the lines

$$x + 2y = 100,$$
$$2x - y = 0,$$
$$2x + y = 200.$$

Then, $x + 2y = 100$

x	0	100
y	50	0

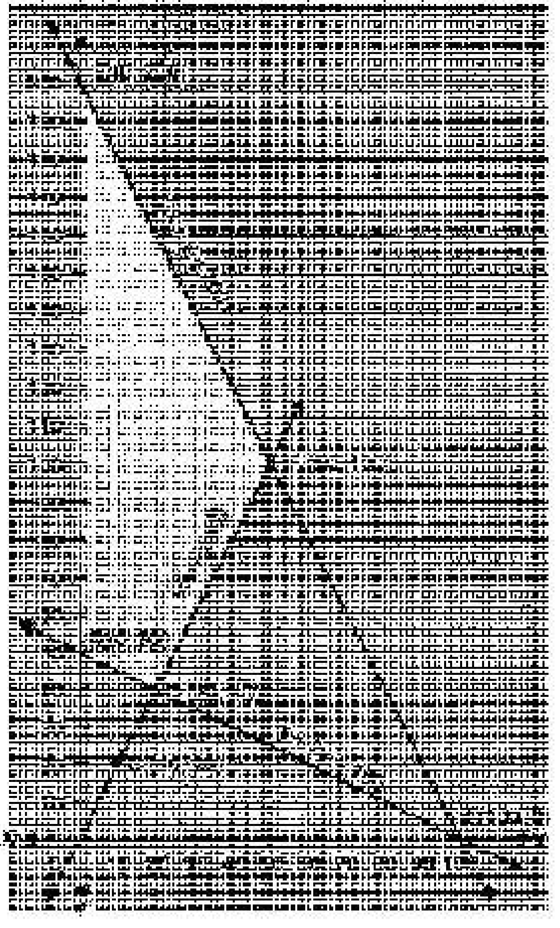
$2x - y = 0$

x	10	20
y	20	40

and $2x + y = 200$

x	0	100
y	200	0

Plotting these points on the graph, we get the shaded feasible region *i.e.*, ADFEA.



Corner points	Value of $Z = x + 2y$
A (0, 50)	$(0) + 2(50) = 100$
D (20, 40)	$20 + 2(40) = 100$
F (50, 100)	$50 + 2(100) = 250$
E (0, 200)	$0 + 2(200) = 400$

Clearly, the maximum value of Z is 400 at (0, 200).

Ans.

SECTION — D

24. Determine the product:

$$\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$ **and use it to solve the system of equations $x - y + z = 4$, $x - 2y - 2z = 9$, $2x + y + 3z = 1$.** **[6]**

Solution :

Let $A = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

Then, $AB = \begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$\Rightarrow \quad AB = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I$

$\Rightarrow \quad AB = 8I$

$\Rightarrow \quad ABB^{-1} = 8IB^{-1}$

$\Rightarrow \quad \frac{1}{8}A = B^{-1}$ ...(i)

Now, consider the given system of equations :

$$x - y + z = 4$$
$$x - 2y - 2z = 9$$
$$2x + y + 3z = 1$$

which can be expressed as $BX = C$,

where, $B = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

$X = B^{-1}C$

$X = \frac{1}{8}AC$ [using (i)]

$\therefore \quad X = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$

$$= \frac{1}{8}\begin{bmatrix} -16+36+4 \\ -28+9+3 \\ 20-27-1 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 24 \\ -16 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$\therefore \quad x = 3, \quad y = -2, \quad z = -1$ **Ans.**

25. Consider $f : R - \left\{-\frac{4}{3}\right\} \to R - \left\{\frac{4}{3}\right\}$ given by $f(x) = \frac{4x+3}{3x+4}$. Show that f is bijective. Find the inverse of f and hence find $f^{-1}(0)$ and x such that $f^{-1}(x) = 2$. **[6]**

Solution : For one-one :

Let $x, y \in R - \left\{-\frac{4}{3}\right\}$

and $f(x) = f(y)$

$$\frac{4x+3}{3x+4} = \frac{4y+3}{3y+4}$$

$\Rightarrow \quad (4x+3)(3y+4) = (3x+4)(4y+3)$

$\Rightarrow \quad 12xy + 16x + 9y + 12 = 12xy + 9x + 16y + 12$

$\Rightarrow \quad 16x + 9y - 9x - 16y = 0$

$\Rightarrow \quad 7x - 7y = 0$

$\Rightarrow \quad 7(x - y) = 0$

$\Rightarrow \quad x - y = 0$

$\Rightarrow \quad x = y$

Hence, f is one-one function.

For onto :

Let $y \in R - \left\{\frac{4}{3}\right\}$, then

$f(x) = y$

$$\Rightarrow \quad \frac{4x+3}{3x+4} = y$$

$$\Rightarrow \quad 4x+3 = 3xy+4y$$

$$\Rightarrow \quad (4-3y)x = 4y-3$$

$$\Rightarrow \quad x = \frac{4y-3}{4-3y} \quad ...(i)$$

As $y \in R - \left\{\frac{4}{3}\right\}, x = \frac{4y-3}{4-3y} \in R$

Also $\frac{4y-3}{4-3y} \neq -\frac{4}{3}$

Since if, $\frac{4y-3}{4-3y} = -\frac{4}{3}$

$$\Rightarrow \quad 12y-9 = -16+12y$$

$$\Rightarrow \quad 9 = 16 \text{ which is not possible.}$$

Thus, $x = \frac{4y-3}{4-3y} \in R - \left\{-\frac{4}{3}\right\}$ such that

$$f(x) = f\left(\frac{4y-3}{4-3y}\right)$$

$$\Rightarrow \quad f(x) = \frac{4\left(\frac{4y-3}{4-3y}\right)+3}{3\left(\frac{4y-3}{4-3y}\right)+4}$$

$$= \frac{16y-12+12-9y}{12y-9+16-12y}$$

$$= \frac{7y}{7}$$

$$\Rightarrow \quad f(x) = y$$

Hence, every element $y \in R - \left\{-\frac{4}{3}\right\}$ has its pre image $x \in R - \left\{-\frac{4}{3}\right\}$

Hence, f is onto.

$\Rightarrow f$ is one-one and onto, so f is invertible.

Now, $f(x) = y$

$$\Rightarrow \quad f^{-1}(y) = x$$

$$\Rightarrow \quad f^{-1}(y) = \frac{4y-3}{4-3y} \quad \text{[from eq. (i)]}$$

$$f^{-1}(0) = \frac{4\times 0-3}{4-3\times 0}$$

$$= \frac{-3}{4}$$

**Anwer is not given due to the change in present syllabus

Also given, $f^{-1}(x) = 2$

$$\Rightarrow \quad \frac{4x-3}{4-3x} = 2$$

$$\Rightarrow \quad 4x-3 = 8-6x$$

$$\Rightarrow \quad 10x = 11$$

$$\Rightarrow \quad x = \frac{11}{10} \quad \textbf{Ans.}$$

OR

Let A = Q × Q and let * be a binary operation on A defined by $(a, b) * (c, d) = (ac, b + ad)$ for $(a, b), (c, d) \in A$. Determine, whether * is commutative and associative. Then, with respect to * on A.**

(i) Find the identity element in A

(ii) Find the invertible elements of A.

26. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube. **[6]**

Solution : Let the length and breadth of the cuboid of square base be x and height be y.

Then, volume of the cuboid $(V) = x^2y$

$$\Rightarrow \quad y = \frac{V}{x^2} \quad ...(i)$$

Now, surface area of cuboid,

$$S = 2(x^2 + yx + yx)$$

$$\Rightarrow \quad S = 2(x^2 + 2xy)$$

$$\Rightarrow \quad S = 2\left(x^2 + 2x\left(\frac{V}{x^2}\right)\right) \quad \text{[using (i)]}$$

$$\Rightarrow \quad S = 2\left(x^2 + \frac{2V}{x}\right)$$

Now, differentiating S w.r.t. x, we get

$$\frac{dS}{dx} = 2\left(2x - \frac{2V}{x^2}\right)$$

For maxima and minima, we have

$$\frac{dS}{dx} = 0$$

$$\Rightarrow \quad 2\left(2x - \frac{2V}{x^2}\right) = 0$$

$$\Rightarrow \quad 4\left(x - \frac{V}{x^2}\right) = 0$$

$$x = \frac{V}{x^2} \Rightarrow x = \sqrt[3]{V}$$

Now, $\frac{dS}{dx} = 2\left(2x - \frac{2V}{x^2}\right) = 4\left(x - \frac{V}{x^2}\right)$

Again differentiating w.r.t. x, we get

$$\frac{d^2S}{dx^2} = 4\left(1 + \frac{2V}{x^3}\right)$$

$$\Rightarrow \left[\frac{d^2S}{dx^2}\right]_{x=\sqrt[3]{V}} = 4\left(1+\frac{2V}{V}\right)=12>0$$

Thus, S is minimum when $x = \sqrt[3]{V}$.

Putting $x = \sqrt[3]{V}$ in (i), we get

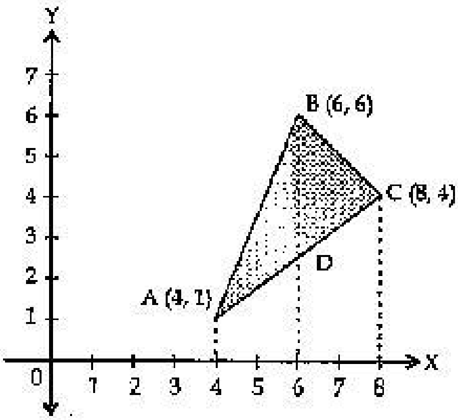
$$y = \frac{V}{x^2} = \frac{x^3}{x^2} = x$$

$$\Rightarrow \quad y = x$$

Hence, it is a cube since the length, breadth and height of a cube are equal. **Hence Proved.**

27. Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are A (4, 1), B (6, 6) and C (8, 4). [6]

Solution : We have, A(4, 1), B (6, 6) and C(8, 4) as the vertices of a triangle ABC.



Then, equation of AB is

$$\Rightarrow \quad \frac{x-4}{6-4} = \frac{y-1}{6-1}$$

$$\Rightarrow \quad \frac{x-4}{2} = \frac{y-1}{5}$$

$$\Rightarrow \quad 5x-20 = 2y-2$$

$$\Rightarrow \quad 5x-2y-18 = 0$$

$$\Rightarrow \quad y = \frac{5x-18}{2} \quad ...(i)$$

Equation of BC is,

$$\frac{x-6}{8-6} = \frac{y-1}{6-1}$$

$$\Rightarrow \quad \frac{x-6}{2} = \frac{y-6}{-2}$$

$$\Rightarrow \quad -2x+12 = 2y-12$$

$$\Rightarrow \quad -2x-2y+24 = 0$$

$$\Rightarrow \quad x+y-12 = 0$$

$$\Rightarrow \quad y = 12-x \quad ...(ii)$$

and equation of CA is

$$\frac{x-8}{4-8} = \frac{y-4}{1-4}$$

$$\Rightarrow \quad \frac{x-8}{-4} = \frac{y-4}{-3}$$

$$\Rightarrow \quad -3x+24 = -4y+16$$

$$-3x+4y-8 = 0 \quad ...(iii)$$

$$\Rightarrow \quad y = \frac{3x-8}{4}$$

Clearly, Area of ΔABC = Area of trapezium ABDE + Area of trapezium BDFC – Area of trapezium ACFE

Hence, area of ΔABC

$$= \int_4^6\left(\frac{5x-18}{2}\right)dx + \int_6^8(12-x)dx - \int_4^8\left(\frac{3x-8}{4}\right)dx$$

$$= \left[\frac{5x^2}{4}-9x\right]_4^6 + \left[12x-\frac{x^2}{2}\right]_6^8 - \left[\frac{3x^2}{8}-2x\right]_4^8$$

$$= \left(\frac{5(6)^2}{4}-9(6)\right) - \left(\frac{5(4)^2}{4}-9(4)\right) + \left(12(8)-\frac{(8)^2}{2}\right) - \left(12(6)-\frac{(6)^2}{2}\right) \cdot \left(\frac{3(8)^2}{8}-2(8)\right) + \left(\frac{3(4)^2}{8}-2(4)\right)$$

$$= (45-54)-(20-36)+(96-32)-(72-18) -(24-16)+(6-8)$$

$$= -9+16+64-54-8-2$$

$= 7$ sq. units. **Ans.**

OR

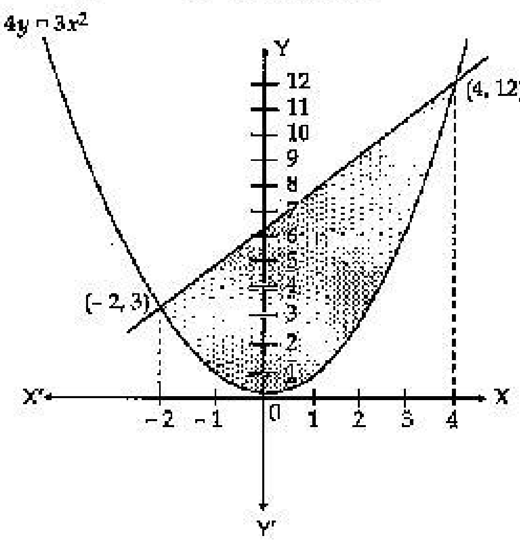
Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

Solution : Given, the equation $4y = 3x^2$ and $3x-2y+12=0$.

$$i.e., \quad y = \frac{3x^2}{4} \quad ...(i)$$

$$\text{and} \quad y = \frac{3x}{2}+6 \quad ...(ii)$$

Solving these equations, we get



$$\frac{3x^2}{4} = \frac{3x}{2}+6$$

$$\Rightarrow \quad 3x^2-6x-24 = 0$$

$$\Rightarrow \quad x^2-2x-8 = 0$$

$$\Rightarrow \quad x^2-4x+2x-8 = 0$$

$$\Rightarrow \quad x(x-4)+2(x-4) = 0$$

$$\Rightarrow \quad (x+2)(x-4) = 0$$

$$\Rightarrow \quad x = -2, 4$$

At $x = -2,\ y = 3$

and at $x = 4,\ y = 12$

The points of intersection are (4, 12) and (– 2, 3).

$$\therefore \text{Required area} = \int_{-2}^{4}\left(\frac{3x}{2}+6\right)dx - \int_{-2}^{4}\frac{3x^2}{4}\,dx$$

$$= \left[\frac{3x^2}{4}+6x\right]_{-2}^{4} - \frac{[x^3]_{-2}^{4}}{4}$$

$$= (12+24-3+12) - \frac{(64+8)}{4}$$

$$= 48-3-18$$

$$= 27 \text{ sq. units.}$$ **Ans.**

28. Find the particular solution of the differential equation $(x-y)\frac{dy}{dx} = (x+2y)$, given that $y = 0$ when $x = 1$. **[6]**

Solution : We have,

$$(x-y)\frac{dy}{dx} = (x+2y)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x+2y}{x-y} \quad \text{...(i)}$$

Putting $y = vx$

and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{x+2vx}{x-vx}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = \frac{1+2v}{1-v}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+2v}{1-v} - v$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+2v-v+v^2}{1-v}$$

$$\frac{1-v}{1+v+v^2}dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\int\frac{1-v}{1+v+v^2}dv = \int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{2}\int\frac{3-1-2v}{1+v+v^2}dv = \int\frac{dx}{x}$$

$$\Rightarrow \quad \frac{3}{2}\int\frac{1}{1+v+v^2}dv - \frac{1}{2}\int\frac{1+2v}{1+v+v^2}dv = \int\frac{dx}{x}$$

$$\Rightarrow \frac{3}{2}\int\frac{dv}{\left(v+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2} - \frac{1}{2}\int\frac{1+2v}{1+v+v^2}dv = \int\frac{dx}{x}$$

$$\Rightarrow \frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) - \frac{1}{2}\log|1+v+v^2| = \log|x| + C$$

$$\Rightarrow \sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) - \frac{1}{2}\log|x^2+xy+y^2| = C \quad \text{...(ii)}$$

Now, given $y = 0$, when $x = 1$

So,

$$\sqrt{3}\tan^{-1}\left(\frac{2(0)+1}{\sqrt{3}}\right) - \frac{1}{2}\log|1+0+0| = C$$

$$\Rightarrow \quad C = \sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\sqrt{3}\pi}{6}$$

Putting the value of C in (ii), we get

$$\sqrt{3}\tan^{-1}\left(\frac{2y+x}{\sqrt{3}x}\right) - \frac{1}{2}\log|x^2+xy+y^2| = \frac{\sqrt{3}\pi}{6}$$

which is the required solution. **Ans.**

29. Find the coordinates of the point where the line through the points (3, – 4, – 5) and (2, – 3, 1), crosses the plane determined by the points (1, 2, 3), (4, 2, – 3) and (0, 4, 3). **[6]**

Solution : Equation of the plane determined by the points (1, 2, 3), (4, 2, – 3) and (0, 4, 3) is

$$\begin{bmatrix} x-1 & y-2 & z-3 \\ 4-1 & 2-2 & -3-3 \\ 0-1 & 4-2 & 3-3 \end{bmatrix} = 0$$

$$\begin{bmatrix} x-1 & y-2 & z-3 \\ 3 & 0 & -6 \\ -1 & 2 & 0 \end{bmatrix} = 0$$

$$(x-1)(0+12) - (y-2)(0-6) + (z-3)(6-0) = 0$$

$$\Rightarrow \quad 12x-12+6y-12+6z-18 = 0$$

$$\Rightarrow \quad 12x+6y+6z-42 = 0$$

$$\Rightarrow \quad 2x + y + z - 7 = 0$$

$$\Rightarrow \quad 2x + y + z = 7 \quad ...(i)$$

Now, equation of line through $(3, -4, -5)$ and $(2, -3, 1)$ is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4} = \frac{z+5}{1+5} = \lambda$$

i.e., $$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda$$

$\therefore$ Coordinates of any point on line is :

$P(-\lambda + 3, \lambda - 4, 6\lambda - 5)$

Now, the point P crosses the plane.

$\therefore$ It satisfies the equation (i) of plane.

$$2(-\lambda + 3) + (\lambda - 4) + (6\lambda - 5) = 7$$

$$\Rightarrow \quad -2\lambda + 6 + \lambda - 4 + 6\lambda - 5 = 7$$

$$\Rightarrow \quad 5\lambda - 3 = 7$$

$$\Rightarrow \quad 5\lambda = 10$$

$$\therefore \quad \lambda = 2$$

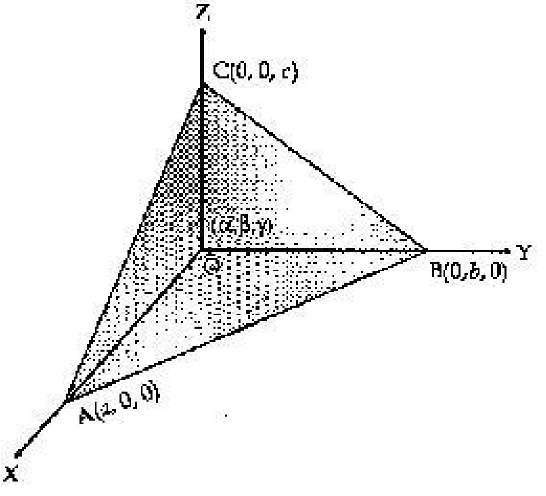
Hence, the point of intersection is $(1, -2, 7)$. **Ans.**

OR

A variable plane which remains at a constant distance $3p$ from the origin cuts the coordinate axes at A, B, C. Show that the locus of the centroid of triangle ABC is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$$

Solution : Let the coordinates of A, B, C are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$



$\therefore$ The equation of plane is :

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad ...(i)$$

Since, the distance of plane is equal to $3p$ from the origin.

Then, $$3p = \frac{\left|\frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1\right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{9p^2} \quad ...(ii)$$

Let the centroid of ΔABC be (x, y, z)

$$= \left(\frac{a+0+0}{3}, \frac{0+b+0}{3}, \frac{0+0+c}{3}\right)$$

$$= \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$$

$$\Rightarrow \quad a = 3x, \quad b = 3y, \quad c = 3z$$

Putting the value of a, b and c in (ii), we get

$$\frac{1}{(3x)^2} + \frac{1}{(3y)^2} + \frac{1}{(3z)^2} = \frac{1}{9p^2}$$

$$\frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2} = \frac{1}{9p^2}$$

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$$

Hence, the required locus of the centroid is :

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}.$$

Hence Proved.

Mathematics 2017 (Outside Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

> **Note : Except for the following questions, all the remaining questions have been asked in previous set.**

SECTION — B

12. The length x, of a rectangle is decreasing at the rate of 5 cm/minute and the width y, is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rate of change of the area of the rectangle. [2]

Solution : We have,

$$\frac{dx}{dt} = -5 \text{ cm/min} \quad ...(i)$$

and $$\frac{dy}{dt} = 4 \text{ cm/min} \quad ...(ii)$$

Now, area of the rectangle, $A = xy$

$$\Rightarrow \quad \frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}$$

$$\frac{dA}{dt} = x(4) + y(-5)$$

[using (i) and (ii)]

$$\therefore \quad \frac{dA}{dt} = 4x - 5y$$

When $x = 8$ cm and $y = 6$ cm,

$$\therefore \quad \left[\frac{dA}{dt}\right]_{\text{at } x=8,\, y=6} = 4(8) - 5(6)$$

$$= 32 - 30 = 2 \text{ cm}^2/\text{min}$$

Hence, the rate of change of the area of the rectangle is2 cm^2/min. **Ans.**

SECTION — C

20. Find : $\int \frac{\sin\theta \, d\theta}{(4+\cos^2\theta)(2-\sin^2\theta)}$. [4]

Solution : Let $I = \int \frac{\sin\theta}{(4+\cos^2\theta)(2-\sin^2\theta)} d\theta$

$$= \int \frac{\sin\theta}{(4+\cos^2\theta)[(2-(1-\cos^2\theta)]} d\theta$$

$$= \int \frac{\sin\theta}{(4+\cos^2\theta)(1+\cos^2\theta)} d\theta$$

Putting $\cos\theta = t$

$\Rightarrow \; -\sin\theta \, d\theta = dt$

$$\therefore \quad I = \int \frac{-dt}{(4+t^2)(1+t^2)}$$

Let $$\frac{-1}{(4+t^2)(1+t^2)} = \frac{A}{4+t^2} + \frac{B}{1+t^2}$$

$$-1 = A(1+t^2) + B(4+t^2)$$

Putting $t = 0$, we get

$$-1 = A + 4B \quad ...(i)$$

Putting $t = 1$, we get

$$-1 = 2A + 5B \quad ...(ii)$$

Solving (i) and (ii), we get

$$A = \frac{1}{3} \text{ and } B = \frac{-1}{3}$$

$$\therefore \quad I = \frac{1}{3}\int \frac{1}{4+t^2} dt - \frac{1}{3}\int \frac{1}{1+t^2} dt$$

$$\Rightarrow \quad I = \frac{1}{3} \times \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) - \frac{1}{3} \times \tan^{-1} t + C$$

$$\Rightarrow \quad I = \frac{1}{6}\tan^{-1}\left(\frac{t}{2}\right) - \frac{1}{3} \times \tan^{-1} t + C$$

$$\Rightarrow \quad I = \frac{1}{6}\tan^{-1}\left(\frac{\cos\theta}{2}\right) - \frac{1}{3} \times \tan^{-1}(\cos\theta) + C$$

Ans.

21. Solve the following linear programming problem graphically :

Maximise $Z = 34x + 45y$

under the following constraints

$$x + y \le 300$$
$$2x + 3y \le 70$$
$$x \ge 0, y \ge 0 \quad [4]$$

Solution : We have,

Maximise $Z = 34x + 45y$

Subject to the constraints :

$$x + y \le 300$$
$$2x + 3y \le 70$$
$$x \ge 0, y \ge 0$$

Converting the given inequalities into equations, we obtain the following equations :

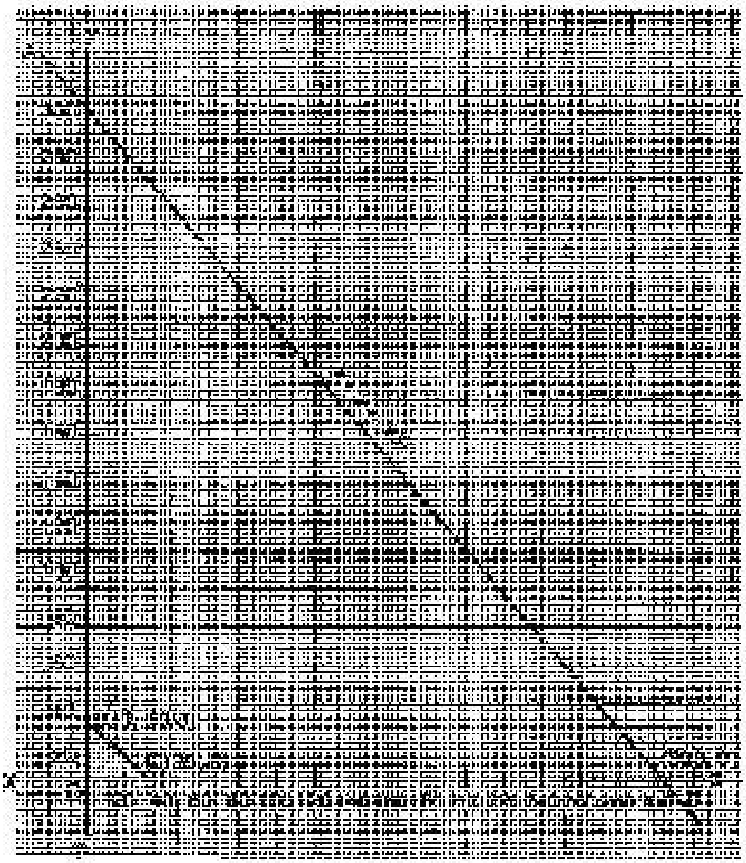
$$x + y = 300$$
$$2x + 3y = 70$$

Then, $x + y = 300$

x	0	300
y	300	0

and $2x + 3y = 70$

x	0	35
y	70/3	0



Plotting these points on the graph, we get the shaded feasible region *i.e.*, OCDO.

Corner point	Value of Z = 34x + 45y
O (0, 0)	34(0) + 45(0) = 0
C (35, 0)	34(35) + 45(0) = 1190
D (0, 70/3)	34(0) + 45 (70/3) = 1050

Clearly, the maximum value of Z is 1190 at (35, 0). **Ans.**

22. Find the value of x such that the points A (3, 2, 1), B (4, x, 5), C(4, 2, – 2) and D (6, 5, – 1) are coplanar. [4]

Solution : Given, the points A(3, 2, 1), B (4, x, 5), C (4, 2, – 2) and D (6, 5, – 1).

$$\therefore \quad \vec{AB} = 4\hat{i} + x\hat{j} + 5\hat{k} - (3\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} + (x-2)\hat{j} + 4\hat{k}$$

and

$$\vec{AC} = 4\hat{i} + x\hat{j} - 2\hat{k} - (3\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - 3\hat{k}$$

$$\vec{AD} = 6\hat{i} + 5\hat{j} - \hat{k} - (3\hat{i} + 2\hat{j} + \hat{k}) = 3\hat{i} + 3\hat{j} - 2\hat{k}$$

Since, the points are coplanar, then

$$\left[\vec{AB}, \vec{AC}, \vec{AD}\right] = 0$$

$$\begin{vmatrix} 1 & x-2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

$$\Rightarrow 1(0+9) - (x-2)(-2+9) + 4(3-0) = 0$$

$$\Rightarrow 9 - 7x + 14 + 12 = 0$$

$$\Rightarrow 7x = 35$$

$$\therefore x = 5 \text{ Ans.}$$

23. Find the general solution of the differential equation :

$$y\,dx - (x + 2y^2)\,dy = 0 \qquad [4]$$

Solution : We have,

$$y dx - (x + 2y^2) dy = 0$$

$$\Rightarrow y dx = (x + 2y^2) dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{x + 2y^2}{y}$$

$$\Rightarrow \frac{dx}{dy} + \left(\frac{-1}{y}\right)x = 2y$$

which is a linear differential equation of the form,

$$\frac{dx}{dy} + Px = Q$$

where $P = \frac{-1}{y}$

and $\quad Q = 2y$

Now, $\quad I.F. = e^{\int P dy} = e^{\int \frac{-1}{y} dy} = e^{\log(1/y)} = \frac{1}{y}$

So, the required solution is

$$x \cdot \frac{1}{y} = \int 2y \cdot \frac{1}{y} dy + C$$

$\therefore \quad \frac{x}{y} = 2y + C$ is the required solution.

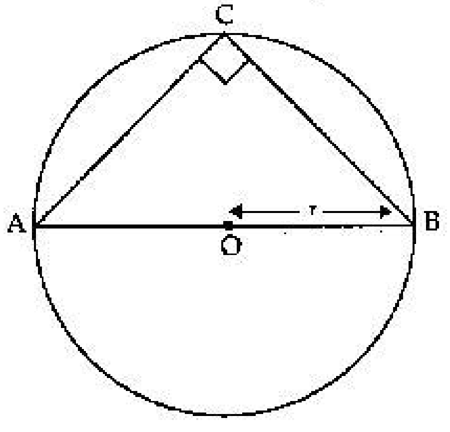
Ans.

SECTION — D

28. AB is the diameter of a circle and C is any point on the circle. Show that the area of triangle ABC is maximum, when it is an isosceles triangle.[6]

Solution : Let r be the radius of the circle then,

$$AB = 2r \quad (\because \text{ AB is diameter})$$



Let, $\quad BC = x$ units

We know that angle subtended by diameter in a circle is right angle

$\therefore \quad \angle C = 90°$

Then, $\quad AC = \sqrt{(AB)^2 - (BC)^2}$

$$AC = \sqrt{(2r)^2 - (x)^2} = \sqrt{4r^2 - x^2} \quad ...(i)$$

Now, area of ΔABC

$$A = \frac{1}{2}(AC)(BC)$$

$$A = \frac{1}{2}\sqrt{4r^2 - x^2}\,(x)$$

Differentiating A w.r.t. x, we get

$$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{2}\left[\sqrt{4r^2 - x^2} + x\frac{1}{2\sqrt{4r^2 - x^2}} \cdot \frac{d}{dx}(4r^2 - x^2)\right]$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{2}\left[\sqrt{4r^2 - x^2} - \frac{x^2}{\sqrt{4r^2 - x^2}}\right]$$

$$= \frac{1}{2}\left[\frac{4r^2 - x^2 - x^2}{\sqrt{4r^2 - x^2}}\right]$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{1}{2}\left[\frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}\right]$$

The critical numbers of x are given by $\frac{dA}{dx} = 0$

$$\Rightarrow \quad \frac{1}{2}\left[\frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}\right] = 0$$

$$\Rightarrow \quad 4r^2 - 2x^2 = 0$$

$$\Rightarrow \quad 4r^2 = 2x^2$$

$$\therefore \quad x = \sqrt{2}r$$

Now, $\quad \frac{dA}{dx} = \frac{1}{2}\left[\frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}\right]$

Again, differentiating w.r.t. x, we get

$$\frac{d^2A}{dx^2} = \frac{1}{2}\left[(-4x)\frac{1}{\sqrt{4r^2 - x^2}} + (4r^2 - 2x^2)\left(\frac{-1}{2}\right)(4r^2 - x^2)^{-3/2}\frac{d}{dx}(4r^2 - x^2)\right]$$

$$\Rightarrow \quad \frac{d^2A}{dx^2} = \frac{1}{2}\left[\frac{-4x}{\sqrt{4r^2 - x^2}} + \frac{x(4r^2 - 2x^2)}{(4r^2 - x^2)^{3/2}}\right]$$

$$\Rightarrow \quad \left(\frac{d^2A}{dx^2}\right)_{x = \sqrt{2}r} = \frac{1}{2}\left[\frac{-4(\sqrt{2}r)}{\sqrt{4r^2 - 2r^2}} + \frac{\sqrt{2}r(4r^2 - 4r^2)}{(4r^2 - 2r^2)^{3/2}}\right]$$

$$= \frac{-2\sqrt{2}r}{\sqrt{2}r} = -2 < 0$$

Thus, A is maximum when $x = \sqrt{2}\,r$.

Putting $x = \sqrt{2}\,r$ in (i),

$$AC = \sqrt{4r^2 - (\sqrt{2}\,r)^2}$$

$$\therefore \quad AC = \sqrt{2}\,r$$

$$\therefore \quad BC = AC = \sqrt{2}\,r$$

Hence, A is maximum when the triangle is isosceles. **Hence Proved.**

29. If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1}. Hence using A^{-1} solve the system of equations $2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$. [6]

Solution : We have,

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$$

$$\therefore \quad |A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$$

$$= 2(-4+4)+3(-6+4)+5(3-2)$$

$$= 2(0)+3(-2)+5(1)$$

$$= -1 \neq 0 .$$

Hence, A is invertible and A^{-1} exists.

Let A_{ij} be the co-factors of elements a_{ij} in $A = [a_{ij}]$. Then,

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -4 \\ 1 & -2 \end{vmatrix} = 0,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} 3 & -4 \\ 1 & -2 \end{vmatrix} = 2$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 1,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -3 & 5 \\ 1 & -2 \end{vmatrix} = -1$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 5 \\ 1 & -2 \end{vmatrix} = -9,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = -5$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -3 & 5 \\ 2 & -4 \end{vmatrix} = 2,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 3 & -4 \end{vmatrix} = 23$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -3 \\ 3 & 2 \end{vmatrix} = 13$$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$$

Now, the given system of equations is expressible as

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

or $\quad AX = B$

where, $A=\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B=\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Now, $\quad AX = B$

$$\Rightarrow \quad A^{-1}AX = A^{-1}B$$

$$\therefore \quad X = A^{-1}B$$

$$\Rightarrow \quad X = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

$$\Rightarrow \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore \quad x = 1, y = 2, z = 3.$ **Ans.**

●●

Mathematics 2017 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

> **Note : Except for the following questions, all the remaining questions have been asked in previous sets.**

SECTION — B

12. The volume of a sphere is increasing at the rate of 8 cm³/s. Find the rate at which its surface area is increasing when the radius of the sphere is 12 cm. **[2]**

Solution : We have,

Volume of sphere $(V) = \frac{4}{3}\pi r^3$

where, r is the radius of sphere.

Now, differentiating V w.r.t. t, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt} = 8 \text{ cm}^3/\text{s}$$

$$\Rightarrow \quad 4\pi r^2\frac{dr}{dt} = 8 \text{ cm}^3/\text{s}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{2}{\pi r^2} \text{cm}^3/\text{s} \qquad \text{....(i)}$$

and surface area of sphere (S) = $4\pi r^2$.

Then, differentiating S w.r.t. t, we get

$$\frac{dS}{dt} = 4\pi \cdot 2r \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dS}{dt} = 8\pi r \cdot \frac{2}{\pi r^2} \qquad \text{[using (i)]}$$

$$\Rightarrow \quad \frac{dS}{dt} = \frac{16}{r} \text{cm}^2/\text{s}.$$

When r = 12 cm,

$$\left[\frac{dS}{dt}\right]_{r=12} = \frac{16}{12} = \frac{4}{3} \text{cm}^2/\text{s}.$$

Hence, the surface area is increasing at the rate of $\frac{4}{3}$ cm^2/s when radius of sphere is 12 cm. **Ans.**

SECTION — C

20. Solve the following linear programming problem graphically : **[4]**

Maximise Z = 7x + 10y

subject to the constraints

$$4x + 6y \le 240$$
$$6x + 3y \le 240$$
$$x \ge 10$$
$$x \ge 0, y \ge 0$$

Solution : We have,

Maximise $Z = 7x + 10y$

Subject to the constraints :

$$4x + 6y \le 240$$
$$6x + 3y \le 240$$
$$x \ge 10$$
$$x \ge 0, y \ge 0$$

Converting the given inequalities into equations, we obtain the following equations :

$$4x + 6y = 240$$
$$6x + 3y = 240$$
$$x = 10$$

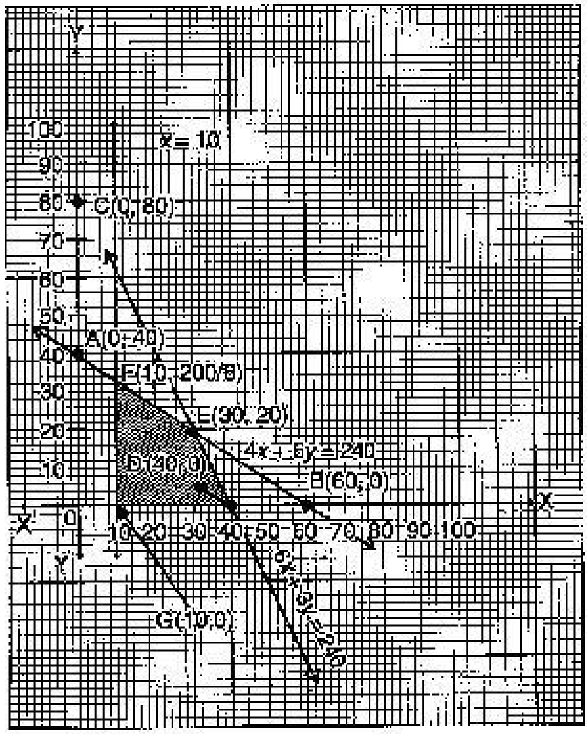
Then, $\quad 4x + 6y = 240$

x	0	60
y	40	0

$6x + 3y = 240$

x	0	40
y	80	0

and x = 10 is a line parallel to Y-axis.



Plotting these points on the graph, we get the shaded feasible region *i.e.*, DEFGD.

Corner points	Value of Z = 7x + 10y
D(40, 0)	7(40) + 10(0) = 280
E (30, 20)	7(30) + 10(20) = 410
F (10, 200/6)	7(10) + 10(200/6) = 403.33
G (10, 0)	7(10) + 10(0) = 70

Clearly the maximum value of Z is 410 at (30, 20).

Ans.

21. Find : $\int \frac{e^x dx}{(e^x - 1)^2 (e^x + 2)}$. **[4]**

Solution : Let $\quad I = \int \frac{e^x dx}{(e^x - 1)^2 (e^x + 2)}$

Putting, $\quad e^x = t \quad$ and $\quad e^x dx = dt$

$$I = \int \frac{dt}{(t-1)^2 (t+2)}$$

Let $\quad \frac{1}{(t-1)^2 (t+2)} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+2}$

$$1 = A(t-1)(t+2) + B(t+2) + C(t-1)^2$$

Putting $t = 1$, we get

$$1 = 3B \quad \Rightarrow \quad B = 1/3$$

Putting $t = -2$, we get

$$1 = 9C \quad \Rightarrow \quad C = 1/9$$

Putting $t = 0$, we get

$$1 = -2A + 2B + C$$

$$\Rightarrow \quad 1 = -2A + \frac{2}{3} + \frac{1}{9}$$

$$\Rightarrow \quad A = -1/9$$

$$\therefore \quad I = \frac{-1}{9}\int \frac{1}{(t-1)}\,dt$$

$$\vec{a}.\vec{b} \int \frac{1}{(t-1)^2}\,dt + \frac{1}{9}\int \frac{1}{(t+2)}\,dt$$

$$\Rightarrow I = \frac{-1}{9}\log|t-1| - \frac{1}{3\cdot(t-1)} + \frac{1}{9}\log|t+2| + C$$

$$\therefore \quad I = \frac{1}{9}\log\left|\frac{e^x+2}{e^x-1}\right| - \frac{1}{3(e^x-1)} + C. \qquad \textbf{Ans.}$$

22. If $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 2\hat{j} - 3\hat{k}$, then express $\vec{b}$ in the form of $\vec{b} = \vec{b_1} + \vec{b_2}$, where $\vec{b_1}$ is parallel to $\vec{a}$ and $\vec{b_2}$ is perpendicular to $\vec{a}$. **[4]**

Solution : We have,

$\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 2\hat{j} - 3\hat{k}$,

Since, $\vec{b_1}$ is parallel to $\vec{a}$

$$\vec{b_1} = \lambda\vec{a}$$

$$\therefore \quad \vec{b_1} = 2\lambda\hat{i} - \lambda\hat{j} - 2\lambda\hat{k}$$

So,

$$\vec{b_2} = \vec{b} - \vec{b_1}$$

$$= 7\hat{i} + 2\hat{j} - 3\hat{k} - (2\lambda\hat{i} - \lambda\hat{j} - 2\lambda\hat{k})$$

$$= (7 - 2\lambda)\hat{i} + (2 + \lambda)\hat{j} + (-3 + 2\lambda)\hat{k}$$

Since, $\vec{b_2}$ is perpendicular to $\vec{a}$

$$\vec{a}\cdot\vec{b_2} = 0$$

$$\therefore (2\hat{i} - \hat{j} - 2\hat{k})[(7-2\lambda)\hat{i} + (2+\lambda)\hat{j} + (-3+2\lambda)\hat{k}] = 0$$

$$\Rightarrow \quad 2(7-2\lambda) - 1(2+\lambda) - 2(-3+2\lambda) = 0$$

$$\Rightarrow \quad 14 - 4\lambda - 2 - \lambda + 6 - 4\lambda = 0$$

$$\Rightarrow \quad -9\lambda + 18 = 0$$

$$\Rightarrow \quad 9\lambda = 18$$

$$\therefore \quad \lambda = 2$$

Hence,

$$\vec{b} = (4\hat{i} - 2\hat{j} - 4\hat{k}) + (3\hat{i} + 4\hat{j} + \hat{k})$$

$$\vec{b} = \vec{b_1} + \vec{b_2} \qquad \textbf{Ans.}$$

23. Find the general solution of the differential equation $\frac{dy}{dx} - y = \sin x$. **[4]**

Solution : We have,

$$\frac{dy}{dx} - y = \sin x$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

where, $P = -1$ and $Q = \sin x$

Now, $\text{I. F.} = e^{\int P dx} = e^{\int -1 dx} = e^{-x}$

So, the required solution is

$$ye^{-x} = \int e^{-x}\sin x\,dx + C_1 \qquad \text{...(i)}$$

Let

$$I = \int e^{-x}\sin x\,dx \qquad \text{...(ii)}$$

$$I = \sin x\int e^{-x}dx - \int\left(\frac{d}{dx}(\sin x)\int e^{-x}\,dx\right)dx + C_2$$

$$\Rightarrow \quad I = -\sin x\,e^{-x} + \int\cos x\,e^{-x}\,dx + C_2$$

$$\Rightarrow \quad I = -\sin x\,e^{-x} + \cos x\int e^{-x}dx - \int\left(\frac{d}{dx}(\cos x)\int e^{-x}dx\right)dx + C_2$$

$$\Rightarrow \quad I = -\sin x\,e^{-x} - \cos x\cdot e^{-x} - \int\sin x\cdot e^{-x}dx + C_2$$

$$\Rightarrow \quad I = -\sin x\cdot e^{-x} - \cos x\cdot e^{-x} - I + C_2 \qquad \text{[using (ii)]}$$

$$\Rightarrow \quad 2I = -e^{-x}(\sin x + \cos x) + C_2$$

$$\Rightarrow \quad I = \frac{-1}{2}e^{-x}(\sin x + \cos x) + C_2$$

By equation (i),

$$ye^{-x} = \frac{-1}{2}e^{-x}(\sin x + \cos x) + C_1 + C_2$$

$$2y = -(\sin x + \cos x) + 2Ce^x$$

$$(\because C_1 + C_2 = C)$$

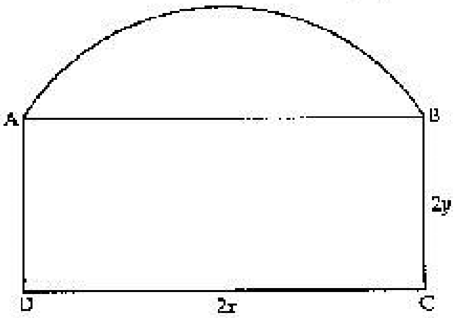
$\therefore$ $2y = 2Ce^x - \sin x - \cos x$ is the required solution. **Ans.**

SECTION – D

29. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening. **[6]**

Solution : Let ABCD be a window of rectangular form surmounted by a semicircle with diameter AB.

Given, Perimeter of the window (P) = 10 m



Let the length and breadth of the rectangle be $2x$ and $2y$ respectively.

Since, $P = 10$ m

i.e., $2x + 4y + \pi x = 10$

$\Rightarrow 4y = 10 - 2x - \pi x$

$\Rightarrow 4xy = 10x - 2x^2 - \pi x^2$ ...(i)

Now, area of the window

$$A = (2x)(2y) + \frac{1}{2}\pi x^2$$

$$\Rightarrow A = 4xy + \frac{1}{2}\pi x^2$$

$$\Rightarrow A = 10x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$$

[using (i)]

$$\Rightarrow A = 10x - 2x^2 - \frac{1}{2}\pi x^2$$

On differentiating A w. r. t. x, we get

$$\frac{dA}{dx} = 10 - 4x - \pi x$$

The critical numbers of x are given by $\frac{dA}{dx} = 0$

$\Rightarrow 10 - 4x - \pi x = 0$

$\Rightarrow -x(4+\pi) = -10$

$\Rightarrow x = \frac{10}{4+\pi}$

Now, $\frac{dA}{dx} = 10 - 4x - \pi x$

differentiating Again, w.r.t. x, we get

$$\frac{d^2A}{dx^2} = -4 - \pi = -(4+\pi) < 0$$

Thus, A is maximum when $x = \frac{10}{4+\pi}$ m

Now, length of the window, $2x = \frac{20}{4+\pi}$ m

and width of the window,

$$2y = \frac{10 - 2x - \pi x}{2} \quad \text{[using (i)]}$$

$$= 5 - \frac{10}{4+\pi} - \frac{\pi}{2}\cdot\frac{10}{4+\pi}$$

$$= \frac{10(4+\pi) - 20 - 10\pi}{2(4+\pi)}$$

$$= \frac{40 + 10\pi - 20 - 10\pi}{2(4+\pi)}$$

$$= \frac{10}{4+\pi} \text{ m}$$

Also, radius of the semi-circle is $\left(\frac{10}{4+\pi}\right)$m

Hence, the dimensions of the rectangular part of the window are $\frac{20}{4+\pi}$ m and $\frac{10}{4+\pi}$ m. **Ans.**

Mathematics 2017 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION — A

1. If A is a 3 × 3 invertible matrix, then what will be the value of k if det $(A^{-1}) = (\det A)^k$. [1]

Solution : Given, A is an invertible matrix.

$\therefore A.A^{-1} = I$

$\Rightarrow \det(A.A^{-1}) = \det(I)$

$\Rightarrow \det(A).\det(A^{-1}) = 1$ [$\because \det(I) = 1$]

$\Rightarrow \det(A).(\det A)^k = 1$ [$\because \det(A^{-1}) = (\det A)^k$]

$$(\det A)^k = \frac{1}{\det(A)}$$

$\therefore (\det A)^k = (\det A)^{-1}$

on comparing both sides, we get

$k = -1$ **Ans.**

2. Determine the value of the constant 'k' so that the function $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$. [1]

Solution : Given, that the function is continuous at $x = 0$

$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$ ...(i)

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{kx}{|x|}$$

$$= k \lim_{x\to 0^-} \frac{x}{-x}$$
$$= k(-1) = -k$$
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3 = 3$$

From equation (i),
$$-k = 3$$
or $$k = -3$$ **Ans.**

3. **Evaluate : $\int_2^3 3^x\,dx$.** **[1]**

Solution : Let $I = \int_2^3 3^x\,dx = \left(\frac{3^x}{\log 3}\right)_2^3 + C$

where C is constant of integration

$$I = \frac{1}{\log 3}\left[3^3 - 3^2\right] + C$$
$$= \frac{1}{\log 3}(27-9) + C$$
$$= \frac{18}{\log 3} + C$$ **Ans.**

4. **If a line makes angles 90° and 60° respectively with the positive directions of X and Y-axes, find the angle which it makes with the positive direction of Z-axis.** **[1]**

Solution : We know that :
$$l^2 + m^2 + n^2 = 1 \quad ...(i)$$
and $l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$

Given, $\alpha = 90°,\ \beta = 60°$

$\therefore \cos\alpha = \cos 90° = 0$ and $\cos\beta = \cos 60° = \frac{1}{2}$

From equation (i),
$$0^2 + \left(\frac{1}{2}\right)^2 + n^2 = 1$$
$$\Rightarrow \quad n^2 = 1 - \frac{1}{4} = \frac{3}{4}$$
$$\Rightarrow \quad \cos^2\gamma = \frac{3}{4}$$
$$\Rightarrow \quad \cos\gamma = \pm\frac{\sqrt{3}}{2}$$
$$\Rightarrow \quad \gamma = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) \text{ or } \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)$$
$$\Rightarrow \quad \gamma = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$ **Ans.**

SECTION – B

5. **Show that all the diagonal elements of a skew symmetric matrix are zero.** **[2]**

Solution : Let $A = [a_{ij}]$ be a given matrix

Since, it is skew symmetric $A' = -A$

$\therefore \quad a_{ji} = -a_{ij}$ For all i, j

$\Rightarrow \quad a_{ii} = -a_{ii}$ For all values of i

$\Rightarrow \quad 2a_{ii} = 0$ For all values of i

$\Rightarrow \quad a_{ii} = 0$ For all values of i

$\Rightarrow \quad a_{11} = a_{22} = a_{33} = \ldots\ldots = a_{nn} = 0$

Hence, all the diagonal elements of a skew. symmetric matrix are zero (as diagonal elements are : $a_{11}; a_{22}; \ldots\ldots a_{nn}$). **Hence Proved.**

6. **Find $\frac{dy}{dx}$ at $x = 1$, $y = \frac{\pi}{4}$ if $\sin^2 y + \cos xy = K$.** **[2]**

Solution : Given, $\sin^2 y + \cos xy = K$

Differentiating both sides w.r. t. x, we get

$$2\sin y\cos y\,.\frac{dy}{dx} + \left[-\sin xy\left(x\frac{dy}{dx} + y\right)\right] = 0$$
$$\Rightarrow \sin 2y\,.\frac{dy}{dx} - x\sin xy\frac{dy}{dx} - y\sin xy = 0$$
$$\Rightarrow \quad \frac{dy}{dx}(\sin 2y - x\sin xy) = y\sin xy$$
$$\frac{dy}{dx} = \frac{y\sin xy}{\sin 2y - x\sin xy} \quad ...(i)$$

Given, at $x = 1, y = \frac{\pi}{4}$

From equation (i),
$$\frac{dy}{dx} = \frac{\frac{\pi}{4}\sin\frac{\pi}{4}}{\sin\frac{2\pi}{4} - 1.\sin\frac{\pi}{4}} = \frac{\frac{\pi}{4}\times\frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}}$$
$$= \frac{\frac{\pi}{4\sqrt{2}}}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{\pi}{4(\sqrt{2}-1)}$$
$$= \frac{\pi\left(\sqrt{2}+1\right)}{4\left(\left(\sqrt{2}\right)^2 - 1^2\right)}$$
$$= \frac{\pi\left(\sqrt{2}+1\right)}{4}$$ **Ans.**

7. **The volume of a sphere is increasing at the rate of 3 cubic centimetre per second. Find the rate of increase of its surface area, when the radius is 2 cm.** **[2]**

Solution : Let V be the volume and r be the radius of sphere at any time t.

Then, $V = \frac{4}{3}\pi r^3$

Given, $\frac{dV}{dt} = 3 \text{ cm}^3/\text{s}$

Differentiating V w.r.t. t, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 . \frac{dr}{dt}$$

$$\Rightarrow \quad 3 = 4\pi r^2 . \frac{dr}{dt}$$

$$\Rightarrow \quad \frac{dr}{dt} = \frac{3}{4\pi r^2} \quad ...(i)$$

Now, let S be the surface area of the sphere,

then $S = 4\pi r^2$

Differentiating S w.r.t. t, we get

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$= 8\pi r \frac{3}{4\pi r^2} \quad \text{[From eq. (i)]}$$

$$= \frac{6}{r}$$

when $r = 2$

$$\left(\frac{dS}{dt}\right)_{r=2} = \frac{6}{2} = 3 \text{ cm}^2/\text{sec}$$

∴ Rate of increase of surface area of the sphere is 3 square centimetre per second. **Ans.**

8. **Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R.** **[2]**

Solution : Given, $f(x) = 4x^3 - 18x^2 + 27x - 7$

Differentiating $f(x)$ w.r.t. x, we get

$$f'(x) = 12x^2 - 36x + 27 = 3(4x^2 - 12x + 9)$$

$$= 3(2x - 3)^2 \quad \text{for any } x \in R$$

$$3 > 0 \text{ and } (2x - 3)^2 \geq 0$$

$$\therefore \quad f'(x) \geq 0$$

⇒ The function is always increasing on R.

Hence Proved.

9. **Find the vector equation of the line passing through the point A(1, 2, – 1) and parallel to the line 5x – 25 = 14 – 7y = 35z.** **[2]**

Solution : Given line is $5x - 25 = 14 - 7y = 35z$

$$\Rightarrow \quad 5(x - 5) = -7(y - 2) = 35z$$

$$\Rightarrow \quad \frac{x-5}{1/5} = \frac{y-2}{-1/7} = \frac{z-0}{1/35}$$

$$\Rightarrow \quad \frac{x-5}{7} = \frac{y-2}{-5} = \frac{z-0}{1}$$

Direction ratios of this line are 7, – 5, 1.

∴ Vector equation of the line which passes through the point A (1, 2, – 1) and its direction ratio are proportional to 7, – 5, 1 is

$$\vec{r} = \hat{i} + 2\hat{j} - \hat{k} + \lambda\left(7\hat{i} - 5\hat{j} + \hat{k}\right) \quad \textbf{Ans.}$$

10. **Prove that if E and F are independent events, then the events E and F′ are also independent.** **[2]**

Solution : Since E and F are independent events :

$$P(E \cap F) = P(E) . P(F) \quad ...(i)$$

$$P(E \cap F') = P(E) - P(E \cap F)$$

$$= P(E) - P(E) P(F) \quad \text{[From (i)]}$$

$$= P(E)(1 - P(F))$$

$$= P(E) P(F')$$

∴ E and F′ are also independent. **Hence Proved.**

11. **A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹ 300. Formulate an L.P.P. for finding how many of each should be produced daily to maximize the profit ? It is being given that at least one of each must be produced.** **[2]**

Solution : Let the manufacturer produces x pieces of necklaces and y pieces of bracelets.

Since total number of necklaces and bracelets that can be handle per day are 24.

so, $x + y \leq 24$ ...(i)

To make bracelet one needs one hour and half an hour is need to make necklace and maximum time available is 16 hours

so, $\frac{1}{2}x + y \leq 16$ ...(ii)

Now, let Z be the profit and we have to maximize it, so our LPP will be

Maximize $Z = 100x + 300y$

Subject to constraints :

$$x + y \leq 24$$

$$\frac{1}{2}x + y \leq 16$$

or $x + 2y \leq 32$

and $x, y \geq 1$ **Ans.**

12. **Find $\int \frac{dx}{x^2 + 4x + 8}$.** **[2]**

Solution : Let $I = \int \frac{dx}{x^2 + 4x + 8}$

$$= \int \frac{dx}{x^2+4x+4-4+8}$$

$$= \int \frac{dx}{(x+2)^2+(2)^2}$$

We know that,

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C,$$

where C is constant of integration

$$\therefore \quad I = \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+C \qquad \textbf{Ans.}$$

SECTION – C

13. Prove that $\tan\left\{\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}$ **[4]**

Solution : Let $\frac{1}{2}\cos^{-1}\frac{a}{b} = A$

$$\text{L.H.S.} = \tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)$$

$$= \frac{\tan\frac{\pi}{4}+\tan A}{1-\tan\frac{\pi}{4}\tan A}+\frac{\tan\frac{\pi}{4}-\tan A}{1+\tan\frac{\pi}{4}\tan A}$$

$$= \frac{1+\tan A}{1-\tan A}+\frac{1-\tan A}{1+\tan A}$$

$$= \frac{(1+\tan A)^2+(1-\tan A)^2}{1-\tan^2 A}$$

$$= \frac{2+2\tan^2 A}{1-\tan^2 A}$$

$$= \frac{2(1+\tan^2 A)}{1-\tan^2 A}$$

$$= 2\times\frac{1}{\cos 2A}$$

$$= 2\times\frac{1}{\cos 2\times\left(\frac{1}{2}\cos^{-1}\frac{a}{b}\right)}$$

$$= 2\times\frac{1}{\cos\left(\cos^{-1}\frac{a}{b}\right)} = \frac{2}{a/b}$$

$$= \frac{2b}{a} = \text{R.H.S} \qquad \textbf{Hence Proved.}$$

14. Using properties of determinants, prove that :

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y). \qquad \textbf{[4]}$$

Solution : Consider, $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x \end{vmatrix}$$

$$= 3(x+y)\begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$, we get

$$= 3(x+y)\begin{vmatrix} 0 & y & y \\ 1 & x & x+y \\ 0 & 2y & -y \end{vmatrix}$$

$$= 3(x+y)[-(-y^2-2y^2)]$$

(expanded along C_1)

$$= 3(x+y)\cdot 3y^2$$

$$= 9y^2(x+y) \qquad \textbf{Hence Proved.}$$

OR

Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$, **find a matrix D such that CD – AB = 0.**

Solution : Let D be the matrix of order 2 × 2,

$$D = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Given, $\quad CD - AB = 0$

$\therefore \quad CD = AB$

$$\begin{bmatrix} 2 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2a+5c & 2b+5d \\ 3a+8c & 3b+8d \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 43 & 22 \end{bmatrix}$$

On comparing both sides, we get

$$2a + 5c = 3 \qquad \text{...(i)}$$

$$2b + 5d = 0$$

$$\Rightarrow \quad b = \frac{-5}{2}d$$

$$3a + 8c = 43 \qquad \text{...(ii)}$$

$$3b + 8d = 22 \qquad \text{...(iii)}$$

Substituting $b = \frac{-5d}{2}$ in equation (iii), we get

$$3\left(\frac{-5d}{2}\right)+8d = 22$$

$$\Rightarrow \quad \frac{-15d}{2}+8d = 22$$

$$\Rightarrow \quad -15d+16d = 44$$

$$\Rightarrow \quad d = 44$$

Also, $$b = \frac{-5}{2}d$$

$$= \frac{-5}{2}\times 44 = -110$$

From equation (i),

$$a = \frac{3-5c}{2}$$

Substituting in equation (ii), we get

$$3\left(\frac{3-5c}{2}\right)+8c = 43$$

$$\Rightarrow \quad 9-15c+16c = 86$$

$$\Rightarrow \quad c = 77$$

and $$a = \frac{3-5\times 77}{2} = \frac{3-385}{2}$$

$$= \frac{-382}{2} = -191$$

$$\therefore \quad D = \begin{bmatrix} -191 & -110 \\ 77 & 44 \end{bmatrix}$$ **Ans.**

15. Differentiate the function $(\sin x)^x + \sin^{-1}\sqrt{x}$ with respect to x. **[4]**

Solution : Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$

Let, $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$\therefore \quad y = u + v$

Differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} \quad ...(i)$$

$$u = (\sin x)^x$$

Taking log on both sides,

$$\log u = \log (\sin x)^x = x \log \sin x$$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = \frac{x}{\sin x}\cos x + \log \sin x$$

$$\Rightarrow \quad \frac{du}{dx} = u\,[x \cot x + \log \sin x]$$

$$= (\sin x)^x\,[x \cot x + \log \sin x] \quad ...(ii)$$

Also $v = \sin^{-1}\sqrt{x}$

Differentiating v w.r.t. x, we get

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}}\times\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}} \quad ...(iii)$$

From equations (i), (ii), and (iii)

$$\frac{dy}{dx} = (\sin x)^x\,[x \cot x + \log \sin x] + \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$\frac{dy}{dx} = (\sin x)^x[x \cot x + \log \sin x] + \frac{1}{2\sqrt{x-x^2}}$$ **Ans.**

OR

If $x^m y^n = (x+y)^{m+n}$, prove that $\frac{d^2y}{dx^2} = 0$.

Solution : Given, $x^m y^n = (x+y)^{m+n}$

Taking log on both sides, we get

$$\log x^m y^n = \log (x+y)^{m+n}$$

$$\Rightarrow \quad m \log x + n \log y = (m+n)\log (x+y)$$

Differentiating both sides w.r.t. x, we get

$$\frac{m}{x}+\frac{n}{y}\cdot\frac{dy}{dx} = \frac{m+n}{(x+y)}\cdot\left[1+\frac{dy}{dx}\right]$$

$$\Rightarrow \quad \frac{m}{x}+\frac{n}{y}\cdot\frac{dy}{dx} = \frac{m+n}{x+y}+\frac{m+n}{x+y}\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(\frac{n}{y}-\frac{m+n}{x+y}\right) = \frac{m+n}{x+y}-\frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{nx+ny-my-ny}{y(x+y)}\right) = \frac{mx+nx-mx-my}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}\left(\frac{nx-my}{nx-my}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{x\frac{dy}{dx}-y}{x^2}$$

$$= \frac{x\times\frac{y}{x}-y}{x^2}$$

$$= \frac{y-y}{x^2} = 0$$ **Hence Proved.**

16. Find $\int \frac{2x}{(x^2+1)(x^2+2)^2}dx$ **[4]**

Solution :

Let $$I = \int \frac{2x}{(x^2+1)(x^2+2)^2}dx$$

Put $x^2 = t$

$\Rightarrow \quad 2x\,dx = dt$

$$I = \int \frac{dt}{(t+1)(t+2)^2}$$

Let

$$\frac{1}{(t+1)(t+2)^2} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{C}{(t+2)^2} \quad \text{...(A)}$$

$$\Rightarrow \frac{1}{(t+1)(t+2)^2} = \frac{A(t+2)^2 + B(t+1)(t+2) - C(t+1)}{(t+1)(t+2)^2}$$

$$1 = A(t+2)^2 + B(t+1)(t+2) + C(t+1)$$

Equating coefficient of t^2, t and constant terms on both sides, we get

$$A + B = 0 \quad \text{...(i)}$$

$$4A + 3B + C = 0 \quad \text{...(ii)}$$

$$4A + 2B + C = 1 \quad \text{...(iii)}$$

Subtracting equation (iii) from (ii), we get

$$B = -1$$

Substituting $B = -1$ in equation (i),

$$A = 1$$

Substituting the values of A and B in (ii), we get

$$4 - 3 + C = 0$$

$$\Rightarrow \quad C = -1$$

From equation (A),

$$\frac{1}{(t+1)(t+2)^2} = \frac{1}{t+1} + \left(\frac{-1}{t+2}\right) + \left(\frac{-1}{(t+2)^2}\right)$$

$$\Rightarrow \int \frac{1}{(t+1)(t+2)^2}\,dt$$

$$= \left(\frac{-x\cos\pi x}{\pi}\right)_0^1 - \int_0^1 -\frac{\cos\pi x}{\pi}\,dx - \int \frac{1}{(t+2)^2}\,dt$$

$$= \log(t+1) - \log(t+2) + \frac{1}{t+2} + C$$

where C is constant of integration.

$$\therefore \int \frac{2x\,dx}{(x^2+1)(x^2+2)^2} = \log\left|\frac{x^2+1}{x^2+2}\right| + \frac{1}{x^2+2} + C$$

Ans.

17. Evaluate : $\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$ **[4]**

Solution : Let, $I = \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx \quad \text{...(i)}$

$$= \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^\pi \frac{(\pi - x).\sin x}{1+\cos^2 x} \quad \text{...(ii)}$$

Adding equations (i) and (ii), we get

$$2I = \pi\int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$$

Put $\cos x = t$

$\Rightarrow \quad -\sin x\,dx = dt$

when $\quad x = 0, \quad t = 1$

and when $x = \pi$, $t = -1$

$$2I = -\pi\int_1^{-1} \frac{dt}{1+t^2}$$

$$= -\pi\int_{-1}^{1} \frac{dt}{1+t^2} \qquad \left[\because \int_a^b f(x)dx = \int_b^a f(x)dx\right]$$

$$\Rightarrow 2I = \pi.\left(\tan^{-1} t\right)_{-1}^{1} = \pi[\tan^{-1}1 - \tan^{-1}(-1)]$$

$$\Rightarrow 2I = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \frac{\pi^2}{2}$$

$$\Rightarrow \quad I = \frac{\pi^2}{4}$$

Ans.

OR

Evaluate : $\int_0^{3/2} |x\sin\pi x|\,dx.$

Solution :

Let $I = \int_0^{3/2} |x\sin\pi x|\,dx.$

$$|x\sin\pi x| = \begin{cases} x\sin\pi x, & 0 \le x \le 1 \\ -x\sin\pi x, & 1 \le x \le \dfrac{3}{2} \end{cases}$$

$$I = \int_0^1 (x\sin\pi x)dx + \int_1^{3/2} (x\sin\pi x)dx$$

$$= \int_0^1 (x\sin\pi x)dx - \int_1^{3/2} (x\sin\pi x)\,dx$$

Applying by parts on both the integrals, we get

$$I = \left(\frac{-x\cos\pi x}{\pi}\right)_0^1 - \int_0^1 -\frac{\cos\pi x}{\pi}\,dx$$

$$-\left[\left(\frac{-x\cos\pi x}{\pi}\right)_1^{3/2} - \int_1^{3/2} -\frac{\cos\pi x}{\pi}\,dx\right]$$

$$I = \frac{1}{2}\int \frac{3-1-2v}{1+v+v^2}\,dv = \int \frac{dx}{x}$$

$$-\left[\left(\frac{\frac{-3}{2}\cos\frac{3\pi}{2} + \cos\pi}{\pi}\right) + \frac{1}{\pi^2}(\sin\pi x)_1^{3/2}\right]$$

$$= \frac{1}{\pi} + \frac{1}{\pi^2}(\sin \pi - \sin 0) - \left[\frac{0-1}{\pi} + \frac{1}{\pi^2}\left(\sin \frac{3\pi}{2} - \sin \pi\right)\right]$$

$$= \frac{1}{\pi} + \frac{1}{\pi^2}(0 - 0) + \frac{1}{\pi} - \frac{1}{\pi^2}(-1-0) = \frac{2}{\pi} + \frac{1}{\pi^2}$$

$$\Rightarrow I = \frac{2\pi + 1}{\pi^2}$$ **Ans.**

18. Prove that $x^2 - y^2 = c(x^2 + y^2)^2$ is the general solution of the differential equation $(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$, where c is a parameter. [4]

Solution : Given differential equation is,

$$(x^3 - 3xy^2)\,dx = (y^3 - 3x^2y)\,dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \quad ...(i)$$

Clearly, it is a homogeneous differential equation.

Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation (i),

$$v + x\frac{dv}{dx} = \frac{x^3 - 3x.(vx)^2}{(vx)^3 - 3x^2.(vx)} = \frac{x^3(1 - 3v^2)}{x^3(v^3 - 3v)}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v} - v$$

$$= \frac{1 - 3v^2 - v^4 + 3v^2}{v^3 - 3v}$$

$$= \frac{1 - v^4}{v^3 - 3v}$$

$$\Rightarrow \frac{v^3 - 3v}{1 - v^4}dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\int \frac{v^3 - 3v}{1 - v^4}dv = \int \frac{dx}{x}$$

$$\int \frac{v^3}{1 - v^4}dv - 3\int \frac{v}{1 - v^4}dv = \int \frac{dx}{x} \quad ...(ii)$$

In the first integral (For L.H.S.)

Put $1 - v^4 = t$

$$\Rightarrow -4v^3 dv = dt,$$

$$\Rightarrow v^3\,dv = \frac{dt}{-4}$$

In the second integral, put $v^2 = z$

$$2v\,dv = dz \Rightarrow v\,dv = \frac{dz}{2}$$

From equation (ii), we get

$$\frac{-1}{4}\int \frac{dt}{t} - \frac{3}{2}\int \frac{dz}{1 - z^2} = \int \frac{dx}{x}$$

$$\Rightarrow \frac{-1}{4}\log t - \frac{3}{2} \times \frac{1}{2}\log \frac{1+z}{1-z} = \log x + \log A$$

$$\left[\because \int \frac{1}{a^2 - x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C\right]$$

where log A is constant of integration.

$$\frac{3}{4}\log \frac{1-z}{1+z} - \frac{1}{4}\log t = \log Ax$$

$$\Rightarrow \frac{1}{4}\log\left|\left(\frac{1-z}{1+z}\right)^3 \frac{1}{t}\right| = \log Ax$$

$$\Rightarrow \log\left(\frac{\left(1 - \frac{y^2}{x^2}\right)^3}{\left(1 + \frac{y^2}{x^2}\right)^3\left(1 - \frac{y^4}{x^4}\right)}\right)^{1/4} = \log Ax$$

$$\Rightarrow \log\left(\frac{\frac{(x^2 - y^2)^3}{(x^2)^3}}{\frac{(x^2 + y^2)^3}{(x^2)^3}\cdot\frac{(x^4 - y^4)}{x^4}}\right)^{1/4} = \log Ax$$

$$\Rightarrow \left(\frac{(x^2 - y^2)^3\,x^4}{(x^2 + y^2)^3.(x^2 - y^2)(x^2 + y^2)}\right)^{1/4} = Ax$$

$$\Rightarrow \left(\frac{(x^2 - y^2)^2\,x^4}{(x^2 + y^2)^4}\right)^{1/4} = Ax$$

$$\Rightarrow \frac{(x^2 - y^2)^{1/2}\,x}{(x^2 + y^2)} = Ax$$

$$\Rightarrow (x^2 - y^2)^{1/2} = A(x^2 + y^2)$$

On squaring both sides, we get

$$x^2 - y^2 = A^2(x^2 + y^2)^2$$

or $$x^2 - y^2 = c(x^2 + y^2)^2,$$

[where $A^2 = C$]

$x^2 - y^2 = c(x^2 + y^2)^2$ is the solution of given differential equation **Ans.**

19. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then :

(a) Let $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

(b) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar. [4]

Solution : Given, $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

(a) When $c_1 = 1$ and $c_2 = 2$

$$\vec{c} = \hat{i} + 2\hat{j} + c_3\hat{k}$$

We know that, $\vec{a}, \vec{b}, \vec{c}$ are coplanar if

$$[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}.(\vec{b} \times \vec{c}) = 0 \quad ...(i)$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 1 & 2 & c_3 \end{vmatrix} = -c_3\hat{j} + 2\hat{k}$$

Given, $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

$$\vec{a}.(\vec{b} \times \vec{c}) = (\hat{i} + \hat{j} + \hat{k}).(-c_3\hat{j} + 2\hat{k}) = 0$$

$\Rightarrow \quad -c_3 + 2 = 0$

$\Rightarrow \quad c_3 = 2.$ **Ans.**

(b) When $c_2 = -1$ and $c_3 = 1$

$$\vec{c} = c_1\hat{i} - \hat{j} + \hat{k}$$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ c_1 & -1 & 1 \end{vmatrix} = -\hat{j} - \hat{k}$$

From equation (i),

$$(\hat{i} + \hat{j} + \hat{k}).(-\hat{j} - \hat{k}) = 0$$

$\Rightarrow -1 - 1 = 0$, which is not possible

$\Rightarrow$ No value of c_1 can make $\vec{a}, \vec{b}$ and $\vec{c}$ coplanar. **Hence Proved.**

20. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a} + \vec{b} + \vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$. **[4]**

Solution : Let $|\vec{a}| = |\vec{b}| = |\vec{c}| = \lambda$ ...(i)

Now, $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular

We have, $\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$ ...(ii)

$$\therefore \quad (\vec{a} + \vec{b} + \vec{b})^2 = (\vec{a} + \vec{b} + \vec{c}).(\vec{a} + \vec{b} + \vec{c})$$

$$= \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{a}.\vec{c} + \vec{b}.\vec{a} + \vec{b}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} + \vec{c}.\vec{b} + \vec{c}.\vec{c}$$

$$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 = 3\lambda^2 \text{ [using (i)]}$$

$$\Rightarrow \quad |\vec{a} + \vec{b} + \vec{c}| = \sqrt{3}\lambda$$

Suppose $\vec{a} + \vec{b} + \vec{c}$ makes angles $\theta_1, \theta_2, \theta_3$ with $\vec{a}, \vec{b}$ and $\vec{c}$ respectively, then

$$(\vec{a} + \vec{b} + \vec{c}).\vec{a}) = |\vec{a} + \vec{b} + \vec{c}||\vec{a}|\cos\theta_1$$

$$\Rightarrow \vec{a}.\vec{a} + \vec{b}.\vec{a} + \vec{c}.\vec{a} = \sqrt{3}\lambda.\lambda\cos\theta_1$$

$$\Rightarrow \quad |\vec{a}|^2 = \sqrt{3}\lambda^2\cos\theta_1,$$

[using (ii)]

$$\Rightarrow \quad \lambda^2 = \sqrt{3}\lambda^2\cos\theta_1$$

$$\Rightarrow \quad \cos\theta_1 = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

Similarly, $\quad \theta_2 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

and $\quad \theta_3 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\therefore \quad \theta_1 = \theta_2 = \theta_3$

Hence, $\vec{a} + \vec{b} + \vec{c}$ is equally inclined with $\vec{a}, \vec{b}$ and $\vec{c}$. **Hence Proved.**

21. The random variable X can take only the values 0, 1, 2, 3. Given that P (X = 0) = P (X = 1) = p and P(X = 2) = P(X = 3) such that $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$, find the value of p. **[4]**

Solution: Given, P(X = 0) = P (X = 1) = p and P (X = 2) = P (X = 3)

Let $\quad$ P (X = 2) = P(X = 3) = k

X	0	1	2	3
P(x)	p	p	k	k

also given that $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$

$\Rightarrow \quad 0 + p + 4k + 9k = 2(0 + p + 2k + 3k)$

$\Rightarrow \quad p + 13k = 2(p + 5k)$

$\Rightarrow \quad 3k = p$ ...(i)

also we know that $\Sigma p_i = 1$

$\Rightarrow \quad p + p + k + k = 1$

$\Rightarrow \quad 2p + 2k = 1$

$\Rightarrow \quad 6k + 2k = 1 \quad$ [using (i)]

$\Rightarrow \quad 8k = 1$

or $\quad k = \frac{1}{8}$

From equation (i), $p = \frac{3}{8}$ **Ans.**

22. Often it is taken that a truthful person commands, more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually a six.

Do you also agree that the value of truthfulness leads to more respect in the society ? [4]

Solution : Let E_1, E_2 and A be the events defined as follows :

E_1 = Six appears on throwing a die.

E_2 = Six does not appear on throwing a die.

and A = the man reports that it is a six

We have,

$$P(E_1) = \frac{1}{6}$$

$$P(E_2) = \frac{5}{6}$$

Now $P(A/E_1)$ = Probability that the man reports that there is a six on the die given that six has occurred on the die = 4/5 (probability that the man speaks truth)

and $P(A/E_2)$ = Probability that the man reports that there is a six on the die given that six has not occurred on the die (probability that the man does not speak truth).

$$= 1 - \frac{4}{5} = \frac{1}{5}$$

By Bayes' theorem, we have

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2)}$$

$$= \frac{\frac{1}{6} \times \frac{4}{5}}{\frac{1}{6} \times \frac{4}{5} + \frac{5}{6} \times \frac{1}{5}}$$

$$= \frac{4}{4+5} = \frac{4}{9}$$

Yes, truthfulness always leads to more respect in the society as truth always wins. **Ans.**

23. Solve the following L.P.P. graphically :

Minimise $Z = 5x + 10y$

Subject to constraints

$$x + 2y \le 120$$
$$x + y \ge 60$$
$$x - 2y \ge 0$$

and $\quad x, y \ge 0$ **[4]**

Solution : We have,

Minimise $Z = 5x + 10y$

Subject to the constraints :

$$x + 2y \le 120$$
$$x + y \ge 60$$
$$x - 2y \ge 0$$

and $\quad x, y \ge 0$

Converting the given inequalities into equations, we obtain the following equations :

$$x + 2y = 120$$
$$x + y = 60$$
$$x - 2y = 0$$

Then, $\quad x + 2y = 120$

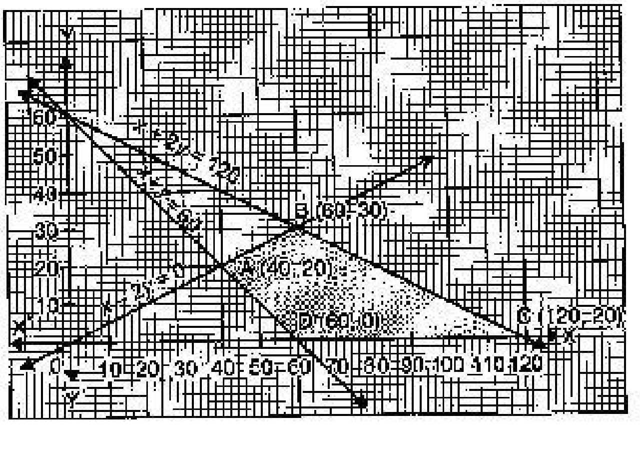
x	0	120
y	60	0

$x + y = 60$

x	0	60
y	60	0

$x - 2y = 0$

x	20	60
y	10	30



The shaded region ABCD represented by the given constraints is the feasible region. Corner points of the common shaded region are

A (40, 20), B (60, 30), C (120, 0) and D (60, 0).
Value of Z at each corner point is given as :

Corner Point	$Z = 5x + 10y$
A (40, 20)	$200 + 200 = 400$
B (60, 30)	$300 + 300 = 600$
C (120, 0)	$600 + 0 = 600$
D (60, 0)	$300 + 0 = 300 \leftarrow$ Minimum

Hence, minimum value of Z is 300 at (60, 0).

Ans.

SECTION – D

24. Use product $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$ **to solve the system of equations** $x + 3z = 9, -x + 2y - 2z = 4, 2x - 3y + 4z = -3.$ **[6]**

Solution : Consider,

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 0+18-18 & 0+4-3 & 0-6+6 \\ -6-18+24 & 0-4+4 & 3+6-8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$

Now, given system of equations can be written in matrix form as follows :

$$\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

$$AX = B$$

$$\Rightarrow \quad X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & -2 \\ 2 & -3 & 4 \end{bmatrix}^{-1}\begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

$$= \left[\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{T}\right]^{-1}\begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

$$= \left[\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}^{-1}\right]^{T}\begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}^{T}\begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 9 & 6 \\ 0 & 2 & 1 \\ 1 & -3 & -2 \end{bmatrix}\begin{bmatrix} 9 \\ 4 \\ -3 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -18+36-18 \\ 0+8-3 \\ 9-12+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$$

$\Rightarrow \quad x = 0, \; y = 5, \; z = 3$ **Ans.**

25. Consider $f : R_+ \to [-5, \infty)$, **given by** $f(x) = 9x^2 + 6x - 5$. **Show that** f **is invertible with** $f^{-1}(y) = \left(\dfrac{\sqrt{y+6} - 1}{3}\right)$. **[6]**

Hence find :

(i) $f^{-1}(10)$

(ii) y **if** $f^{-1}(y) = \dfrac{4}{3}$,

where R_+ **is the set of all non-negative real numbers.**

Solution : For one-one :

Let $x_1, x_2 \in R_+$

$$f(x_1) = f(x_2)$$

$$\Rightarrow \quad 9x_1^2 + 6x_1 - 5 = 9x_2^2 + 6x_2 - 5$$

$$\Rightarrow 9(x_1^2 - x_2^2) + 6(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)(9(x_1 + x_2) + 6) = 0$$

Since $9(x_1 + x_2) + 6 > 0$ $\quad$ [as $x_1, x_2 \in R_+$]

$\Rightarrow \quad x_1 - x_2 = 0$ $\quad$ [$x_1, x_2 \in R_+$]

or $\quad x_1 = x_2$

$\therefore$ Function is one-one

For onto :

For every $y \in [-5, \infty)$ such that $f(x) = y$

$$\Rightarrow \quad 9x^2 + 6x - 5 = y$$

$$\Rightarrow (3x)^2 + 2 \cdot (3x) \cdot 1 + 1^2 - 1^2 - 5 = y$$

$$\Rightarrow \quad (3x+1)^2 - 6 = y$$

$$\Rightarrow \quad (3x+1)^2 = y + 6$$

$$\Rightarrow \quad 3x + 1 = \sqrt{y+6}$$

$$\Rightarrow \quad x = \frac{-1 + \sqrt{y+6}}{3} \in R_+$$

$\therefore$ Function is onto.

Since f is both one-one and onto.

∴ Function is invertible.

$$f(x) = y$$

$$\Rightarrow \quad x = f^{-1}(y) = \frac{-1+\sqrt{y+6}}{3}$$

(i) $$f^{-1}(10) = \frac{-1+\sqrt{16}}{3} = \frac{-1+4}{3} = 1$$

Ans.

(ii) $$f^{-1}(y) = \frac{4}{3}$$

$$\Rightarrow \quad \frac{4}{3} = \frac{-1+\sqrt{y+6}}{3}$$

$$\Rightarrow \quad 4 = -1+\sqrt{y+6}$$

$$\Rightarrow \quad 5 = \sqrt{y+6}$$

Squaring on both sides,

$$25 = y + 6$$

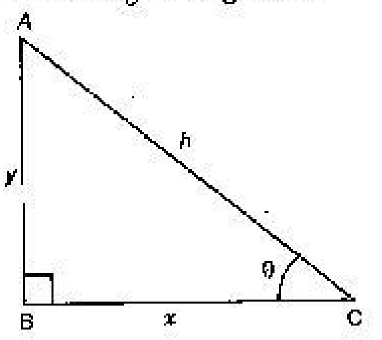
or $$y = 19$$ **Ans.**

OR

Discuss the commutativity and associativity of binary operation '*' defined on A = Q – {1} by the rule $a * b = a - b + ab$ for all, $a, b \in$ A. Also find the identity element of * in A and hence find the invertible elements of A.**

26. If the sum of lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$. [6]

Solution : Let h, x and y be the length of hypotenuse and sides of the right triangle ABC.



From ΔABC

$$h^2 = x^2 + y^2$$

$$\Rightarrow \quad y^2 = h^2 - x^2$$

If A be the area of the triangle ABC, then

$$A = \frac{1}{2}xy = \frac{1}{2}x\sqrt{h^2 - x^2}$$

$$\Rightarrow \quad A^2 = \frac{x^2}{4}\cdot(h^2 - x^2)$$

Let $$A^2 = z$$

[also given $h + x = k$ (constant) $h = k - x$]

$$\Rightarrow \quad z = \frac{x^2}{4}((k-x)^2 - x^2)$$

$$= \frac{x^2}{4}(k^2 - 2kx)$$

$$= \frac{k^2x^2 - 2kx^3}{4}$$

Differentiating z w.r.t. x, we get

$$\frac{dz}{dx} = \frac{2k^2x - 6kx^2}{4}$$

$$= \frac{k^2x - 3kx^2}{2}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2z}{dx^2} = \frac{k^2 - 6kx}{2}$$

For maxima or minima

Put $$\frac{dz}{dx} = 0$$

$$\Rightarrow \quad \frac{k^2x - 3kx^2}{2} = 0$$

$$\Rightarrow \quad kx.(k - 3x) = 0$$

$$\Rightarrow \quad k - 3x = 0, \text{ as } x \neq 0$$

or $$x = \frac{k}{3}$$

$$h = k - x = k - \frac{k}{3} = \frac{2k}{3}$$

$$y^2 = h^2 - x^2$$

$$= \frac{4k^2}{9} - \frac{k^2}{9}$$

$$= \frac{3k^2}{9} = \frac{k^2}{3}$$

$$\Rightarrow \quad y = \frac{k}{\sqrt{3}}$$

when $x = \frac{k}{3}$

$$\left(\frac{d^2z}{dx^2}\right)_{x=\frac{k}{3}} = \frac{1}{2}\left(k^2 - 6k \times \frac{k}{3}\right)$$

$$= \frac{-k^2}{2} < 0$$

∴ Area of the triangle is maximum.

From ΔABC

$$\cos\theta = \frac{x}{h} = \frac{k/3}{2k/3} = \frac{1}{2}$$

or $$\theta = \frac{\pi}{3}$$ **Hence Proved.**

27. Using integration, find the area of region bounded by the triangle whose vertices are (− 2, 1), (0, 4) and (2, 3). [6]

Solution : The vertices of the ΔABC are $A(-2, 1)$, $B(0, 4)$ and $C(2, 3)$.

Equation of the side AB is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow \quad y - 1 = \frac{4-1}{0-(-2)}(x-(-2)) = \frac{3}{2}(x+2)$$

$$\Rightarrow \quad y = \frac{3}{2}x + 4$$

Equation of the side BC is

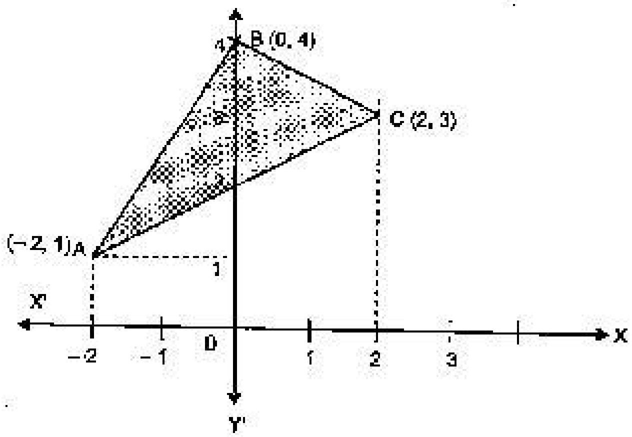
$$y - 4 = \frac{3-4}{2-0}(x-0) = \frac{-1}{2}x$$

$$\Rightarrow \quad y = \frac{-1}{2}x + 4$$

Equation of the side AC is

$$y - 1 = \frac{3-1}{2-(-2)}(x-(-2)) = \frac{1}{2}(x+2)$$

$$\Rightarrow \quad y = \frac{1}{2}x + 2$$



Required area = Shaded area

$$= \int_{-2}^{0}\left(\frac{3}{2}x + 4\right)dx + \int_{0}^{2}\left(\frac{-1}{2}x + 4\right)dx - \int_{-2}^{2}\left(\frac{1}{2}x + 2\right)dx$$

$$= \left[\frac{3}{2}\cdot\frac{x^2}{2} + 4x\right]_{-2}^{0} + \left[\frac{-1}{2}\cdot\frac{x^2}{2} + 4x\right]_{0}^{2} - \left[\frac{1}{2}\cdot\frac{x^2}{2} - 2x\right]_{-2}^{2}$$

$$= (0+0) - (3-8) + (-1+8) - (0+0) - (1+4) + (1-4)$$

$$= 0 + 5 + 7 - 0 - 5 - 3$$

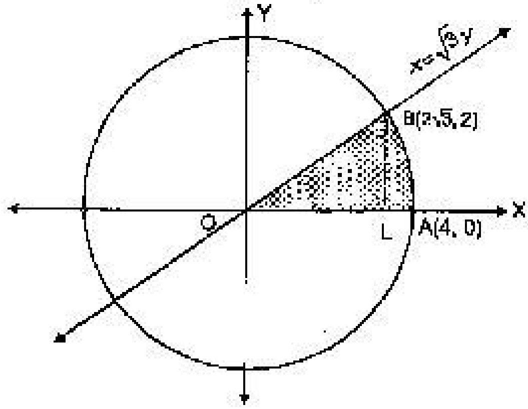
$= 4$ sq. units. **Ans.**

OR

Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $\sqrt{3}y = x$ in the first quadrant, using integration.

Solution : Given, $x = \sqrt{3}y$

and $x^2 + y^2 = 16$,



$$\Rightarrow \quad (\sqrt{3}y)^2 + y^2 = 16$$

$$\Rightarrow \quad 4y^2 = 16$$

$$\Rightarrow \quad y^2 = 4$$

$$\Rightarrow \quad y = 2$$

$$\therefore \quad x = \sqrt{3}y = 2\sqrt{3}$$

$\therefore$ $B(2\sqrt{3}, 2)$ is the point of intersection in first quadrant.

Required area = Area under OBL + Area under LBA

$$= \int_{0}^{2\sqrt{3}} \frac{x}{\sqrt{3}}dx + \int_{2\sqrt{3}}^{4}\sqrt{16 - x^2}\,dx$$

$$= \frac{1}{\sqrt{3}}\left(\frac{x^2}{2}\right)_{0}^{2\sqrt{3}} + \left[\frac{x}{2}\sqrt{16-x^2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_{2\sqrt{3}}^{4}$$

$$= \frac{1}{2\sqrt{3}}(12-0) + \left(0 + 8\sin^{-1}1\right) - \left(\frac{2\sqrt{3}}{2}\sqrt{16-12} + 8\sin^{-1}\frac{2\sqrt{3}}{4}\right)$$

$$= \frac{6}{\sqrt{3}} + 8\times\frac{\pi}{2} - 2\sqrt{3} - 8\sin^{-1}\frac{\sqrt{3}}{2}$$

$$= \frac{6\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - 8\times\frac{\pi}{3}$$

$$= 2\sqrt{3} + \frac{4\pi}{3} - 2\sqrt{3}$$

$= \frac{4\pi}{3}$ sq. units. **Ans.**

28. Solve the differential equation $x\frac{dy}{dx} + y = x\cos x + \sin x$, given that $y = 1$ when $x = \frac{\pi}{2}$. [6]

Solution : Given differential equation is :

$$x\frac{dy}{dx} + y = x\cos x + \sin x$$

$$\Rightarrow \quad \frac{dy}{dx}+\frac{y}{x}=\cos x+\frac{\sin x}{x}$$

which is of the form $\frac{dy}{dx}+Py=Q$,

where $P=\frac{1}{x}$, $Q=\cos x+\frac{\sin x}{x}$

$$I.F.=e^{\int P dx}=e^{\int \frac{1}{x}dx}=e^{\log x}=x$$

Required solution is

$$y \cdot I.F.=\int Q \cdot I.F.+C$$

$$y.x=\int\left(\cos x+\frac{\sin x}{x}\right)x\,dx+C$$

$$=\int x\cos x\,dx+\int \sin x\,dx+C$$

$$=x.\int\cos x\,dx-\int\left[\frac{d}{dx}x.\int\cos x\,dx\right]dx$$

$$-\cos x+C$$

$$\Rightarrow \quad xy=x\sin x-\int \sin x\,dx-\cos x+C$$

$$\Rightarrow \quad xy=x\sin x+\cos x-\cos x+C$$

$$\Rightarrow \quad xy=x\sin x+C \qquad ...(i)$$

Given, $y=1$ when $x=\frac{\pi}{2}$

From equation (i)

$$1\times\frac{\pi}{2}=\frac{\pi}{2}\sin\frac{\pi}{2}+C$$

$$\Rightarrow \quad \frac{\pi}{2}=\frac{\pi}{2}+C$$

$$\Rightarrow \quad C=0$$

Substitute the value of $C=0$ in (i), we get

$$xy=x\sin x$$

$\Rightarrow \quad y=\sin x$, which is the required solution.

Ans.

29. Find the equation of the plane through the line of intersection of $\vec{r}\cdot(2\hat{i}-3\hat{j}+4\hat{k})=1$ and $\vec{r}\cdot(\hat{i}-\hat{j})+4=0$ and perpendicular to the plane $\vec{r}\cdot(2\hat{i}-\hat{j}+\hat{k})+8=0$. Hence find whether the plane thus obtained contains the line $x-1=2y-4=3z-12$. **[6]**

Solution : The equation of the plane passing through the line of intersection of the given planes is :

$$\vec{r}[(2\hat{i}-3\hat{j}+4\hat{k})-1+\lambda(\hat{i}-\hat{j})]+4\lambda-1=0$$

$$\Rightarrow \quad \vec{r}.[(2+\lambda)\hat{i}-(3+\lambda)\hat{j}+4\hat{k}]=1-4\lambda$$

Taking $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, we get

$$(2+\lambda)x-(3+\lambda)y+4z=1-4\lambda \qquad ...(i)$$

$\because$ It is perpendicular to the plane

$$\vec{r}.(2\hat{i}-\hat{j}+\hat{k})+8=0$$

Cartesian equation of this plane is :

$$2x-y+z+8=0 \qquad ...(ii)$$

$\because$ Equations (i) and (ii) are perpendicular

$$\therefore \quad (2+\lambda)2+(3+\lambda)+4=0$$

$$\Rightarrow \quad 4+2\lambda+3+\lambda+4=0$$

$$\Rightarrow \quad 11+3\lambda=0$$

or $\qquad \lambda=\frac{-11}{3}$

From equation (i),

$$\left(2-\frac{11}{3}\right)x-\left(3-\frac{11}{3}\right)y+4z=1-4\times\frac{-11}{3}$$

$$\Rightarrow \quad \frac{-5x}{3}+\frac{2}{3}y+4z=\frac{47}{3}$$

$$\Rightarrow \quad -5x+2y+12z=47$$

Required vector equation of this plane is :

$$\vec{r}.\left(-5\hat{i}+2\hat{j}+12\hat{k}\right)=47 \qquad ...(iii)$$

Now, equation of the given line is,

$$x-1=2y-4=3z-12$$

$$\Rightarrow \quad \frac{x-1}{1}=2(y-2)=3(z-4)$$

$$\Rightarrow \quad \frac{x-1}{1}=\frac{y-2}{1/2}=\frac{z-4}{1/3}$$

$$\Rightarrow \quad \frac{x-1}{6}=\frac{y-2}{3}=\frac{z-4}{2}$$

Vector equation of this line is :

$$\vec{r}=(\hat{i}+2\hat{j}+4\hat{k})+\lambda(6\hat{i}+3\hat{j}+2\hat{k}) \qquad ...(iv)$$

Obviously plane (iii) contains the line (iv) since the point $\hat{i}+2\hat{j}+4\hat{k}$ satisfy the equation of plane (iii) $\left[\text{as } (\hat{i}+2\hat{j}+4\hat{k}).\left(-5\hat{i}+2\hat{j}+12\hat{k}\right)=-5+4+48=47\right]$ and vector $-5\hat{i}+2\hat{j}+12\hat{k}$ is perpendicular to $6\hat{i}+3\hat{j}+2\hat{k}$ as $\left(6\hat{i}+3\hat{j}+2\hat{k}\right)\cdot\left(-5\hat{i}+2\hat{j}+12\hat{k}\right)$

$$=-30+6+24=0$$

$\therefore$ Plane (iii) contains the line (iv). **Ans.**

OR

Find the vector and Cartesian equations of a line passing through (1, 2, – 4) and perpendicular

to the two lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

Solution : Cartesian equation of the line passing through (1, 2, – 4) is

$$\frac{x-1}{a} = \frac{y-2}{b} = \frac{z+4}{c} \quad ...(i)$$

Given lines are

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \quad ...(ii)$$

and $$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad ...(iii)$$

Let $\vec{b_1}, \vec{b_2}, \vec{b_3}$ are parallel vectors of (i), (ii) and (iii) respectively :

$$\vec{b_1} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\vec{b_2} = 3\hat{i} - 16\hat{j} + 7\hat{k}$$

$$\vec{b_3} = 3\hat{i} + 8\hat{j} - 5\hat{k}$$

Given that (i) is perpendicular to both (ii) and (iii)

$\Rightarrow$ $$\vec{b_1} \cdot \vec{b_2} = 0$$

$$3a - 16b + 7c = 0 \quad ...(iv)$$

and $$\vec{b_1} \cdot \vec{b_3} = 0$$

$$3a + 8b - 5c = 0 \quad ...(v)$$

From equations (iv) and (v),

$$\frac{a}{80-56} = \frac{b}{21+15} = \frac{c}{24+48}$$

$$\Rightarrow \quad \frac{a}{24} = \frac{b}{36} = \frac{c}{72}$$

$$\Rightarrow \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{6} = \lambda \text{ (say)}$$

$$\Rightarrow \quad a = 2\lambda, b = 3\lambda, c = 6\lambda$$

Putting in (i),

$$\frac{x-1}{2\lambda} = \frac{y-2}{3\lambda} = \frac{z+4}{6\lambda}$$

$$\Rightarrow \quad \frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

which is the cartesian equation of the line and vector equation of this line is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

Ans.

Mathematics 2017 (Delhi) SET II

Time allowed : 3 hours — Maximum marks : 100

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — B

12. For the curve $y = 5x - 2x^3$, if x increases at the rate of 2 units/sec, then find the rate of change of the slope of the curve when $x = 3$. **[2]**

Solution : Given curve is,

$$y = 5x - 2x^3$$

and $$\frac{dx}{dt} = 2 \text{ units/sec.}$$

Differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = 5 - 6x^2.$$

Differentiating both sides w.r.t. t, we get

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = -12x\frac{dx}{dt}$$

when $x = 3$,

$$\frac{d}{dt}\left(\frac{dy}{dx}\right)_{x=3} = -12 \times 3 \times 2 = -72 \text{ units/sec.}$$

Thus, the slope of the curve is decreasing at the rate of 72 units/sec when $x = 3$. **Ans.**

SECTION — C

20. The random variable X can take only the values 0, 1, 2, 3. Given that P(2) = P(3) = p and P(0) = 2P(1). If $\Sigma p_i x_i^2 = 2\Sigma p_i x_i$, find the value of p. **[4]**

Solution : Given, P(2) = P(3) = p

and P(0) = 2P(1)

Let P(1) = k

X	0	1	2	3
P(x)	2k	k	p	p

also given that $\Sigma p_i x_i^2 = 2\,\Sigma p_i x_i$

$\Rightarrow \quad 0 + k + 4p + 9p = 2(0 + k + 2p + 3p)$

$\Rightarrow \quad k + 13p = 2k + 10p$

or $\quad 3p = k \quad$ ...(i)

also we know that

$\Sigma p_i = 1$

$\Rightarrow \quad 2k + k + p + p = 1$

$\Rightarrow \quad 3k + 2p = 1$

$\Rightarrow \quad 9p + 2p = 1 \quad$ [using (i)]

$\Rightarrow \quad 11p = 1$

or $\quad p = \frac{1}{11}$. **Ans.**

21. Using vectors find the area of triangle ABC with vertices A(1, 2, 3), B(2, – 1, 4) and C (4, 5, – 1). **[4]**

Solution : Vertices of the given ΔABC are A(1, 2, 3) B(2, – 1, 4) and C(4, 5, – 1)

$$\vec{AB} = \vec{OB} - \vec{OA} = (2\hat{i} - \hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= \hat{i} - 3\hat{j} + \hat{k}$$
$$\vec{AC} = \vec{OC} - \vec{OA} = (4\hat{i} + 5\hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k})$$
$$= 3\hat{i} + 3\hat{j} - 4\hat{k}$$

Required area $= \frac{1}{2}|\vec{AB} \times \vec{AC}|$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \end{vmatrix} = 9\hat{i} + 7\hat{j} + 12\hat{k}$$

$$|\vec{AB} \times \vec{AC}| = \sqrt{9^2 + 7^2 + 12^2}$$
$$= \sqrt{81 + 49 + 144} = \sqrt{274}$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}\sqrt{274}$ sq. units **Ans.**

22. Solve the following L. P. P. graphically
Maximise Z = 4x + y

Subject to following constraints :

$$x + y \le 50,$$
$$3x + y \le 90,$$
$$x \ge 10$$
$$x, y \ge 0$$ [4]

Solution : We have,

Maximise $Z = 4x + y$

Subject to the constraints :

$$x + y \le 50$$
$$3x + y \le 90$$
$$x \ge 10$$
$$x, y \ge 0$$

Converting the given inequalities into equations, we obtain the following equations :

$$x + y = 50$$
$$3x + y = 90$$
$$x = 10$$

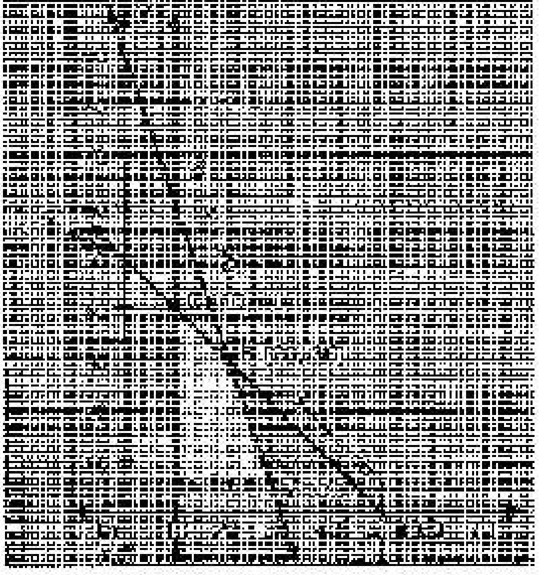
Then, $x + y = 50$

x	0	50
y	50	0

$3x + y = 90$

x	0	30
y	90	0

$x = 10$ is a line which is parallel to Y-axis



Plotting these points on the graph, we get the shaded feasible region *i.e.,* ABCD.

Corner point	Value of $Z = 4x + y$
A (30, 0)	$Z = 4 \times 30 + 0 = 120$ ← Maximum
B (20, 30)	$Z = 4 \times 20 + 30 = 110$
C (10, 40)	$Z = 10 \times 4 + 40 = 80$
D (10, 0)	$Z = 10 \times 4 + 0 = 40$

$\therefore$ Maximum value of Z is 120 at (30, 0) **Ans.**

23. Find : $\int \frac{2x}{(x^2 + 1)(x^4 + 4)} dx$. **[4]**

Solution : Let, $I = \int \frac{2x}{(x^2 + 1)(x^4 + 4)} dx$

$$= \int \frac{2x}{(x^2 + 1)((x^2)^2 + 4)} dx$$

Put $x^2 = t$,

$\Rightarrow \quad 2x dx = dt$

$$I = \int \frac{dt}{(t + 1)(t^2 + 4)}$$

Let $\frac{1}{(t + 1)(t^2 + 4)} = \frac{A}{t + 1} + \frac{Bt + C}{t^2 + 4}$ ...(A)

$$\frac{1}{(t + 1)(t^2 + 4)} = \frac{A(t^2 + 4) + (Bt + C)(t + 1)}{(t + 1)(t^2 + 4)}$$

$$1 = A(t^2+4) + (Bt + C)(t+1)$$

Equating coefficients of t^2, t and constant terms on both sides, we get

$$A + B = 0 \quad \Rightarrow \quad A = -B \qquad ...(i)$$

$$B - C = 0 \quad \Rightarrow \quad B = -C \qquad ...(ii)$$

$$4A + C = 1 \qquad ...(iii)$$

From equations (i) and (ii) equations we get A = C.

From equation (iii)

$$4C + C = 1$$

$$5C = 1$$

or $$C = \frac{1}{5}$$

$$A = C = \frac{1}{5}$$

$$B = \cdot\, C = \frac{-1}{5}$$

From equation (A),

$$\Rightarrow \quad \frac{1}{(t+1)(t^2+4)} = \frac{1/5}{t+1} + \frac{\frac{-1}{5}t + \frac{1}{5}}{t^2+4}$$

$$\Rightarrow \int \frac{1}{(t+1)(t^2-4)}dt = \frac{1}{5}\int \frac{1}{t+1}dt - \frac{1}{5}\int \frac{t}{t^2+4}dt + \frac{1}{5}\int \frac{1}{t^2+4}dt + C$$

Where C is constant of integration.

$$I = \frac{1}{5}\int \frac{1}{t+1}dt - \frac{1}{10}\int \frac{2t}{t^2-4}dt + \frac{1}{5}\int \frac{1}{t^2+2^2}dt + C$$

$$= \frac{1}{5}\log|t+1| - \frac{1}{10}\log|t^2+4| + \frac{1}{5}\times\frac{1}{2}\tan^{-1}\frac{t}{2} + C$$

$$\Rightarrow I = \frac{1}{5}\log|x^2+1| - \frac{1}{10}\log|x^4+4| + \frac{1}{10}\tan^{-1}\frac{x^2}{2} + C$$

Ans.

SECTION — D

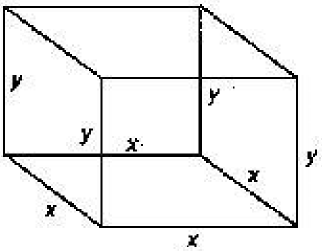
28. A metal box with a square base and vertical sides is to contain 1024 cm^3. The material for the top and bottom costs ₹ 5 per cm^2 and the material for the sides costs ₹2.50 per cm^2. Find the least cost of the box. [6]

Solution : Let the length, breadth and height of the metal box be x cm, x cm and y cm respectively. It is given that,

$$x^2y = 1024$$

$$\Rightarrow \quad y = \frac{1024}{x^2} \qquad ...(i)$$

Let c be the total cost (in rupees) of material used to construct the box.



Then, $$c = 5x^2 + 5x^2 + \frac{5}{2}\times 4xy$$

$$= 10x^2 + 10xy$$

$$= 10x^2 + 10x \times \frac{1024}{x^2}$$

$$= 10x^2 + \frac{10240}{x}$$

Now, differentiating 'c' w.r.t. x, we get

$$\frac{dc}{dx} = 20x - \frac{10240}{x^2}$$

Again, differentiating w.r.t. x, we get

$$\frac{d^2c}{dx^2} = 20 + \frac{20480}{x^3}$$

For maxima or minima

Put $\frac{dc}{dx} = 0$

$$\Rightarrow \quad 20x - \frac{10240}{x^2} = 0$$

$$\Rightarrow \quad x^3 = 512$$

$$\Rightarrow \quad x^3 = 8^3$$

$$\Rightarrow \quad x = 8$$

when $x = 8$

$$\left(\frac{d^2c}{dx^2}\right)_{x=8} = 20 + \frac{20480}{8^3} > 0$$

Thus, the cost of the box is least when $x = 8$, putting $x = 8$ in equation (i) we obtain $y = 16$, so the dimensions of the box are $8 \times 8 \times 16$

Putting, $x = 8$ and $y = 16$ in $c = 10x^2 + 10xy$, we get

$$c = 10 \times 64 + 10 \times 8 \times 16 = 1920$$

Hence, the least cost of the box is ₹ 1920. **Ans.**

29. If $A = \begin{pmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{pmatrix}$, **find A^{-1}. Using A^{-1} solve the system of equations :**

$$\frac{2}{x} + \frac{3}{y} + \frac{10}{z} = 2;$$

$$\frac{4}{x} - \frac{6}{y} + \frac{5}{z} = 5;$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=-4 \qquad \textbf{[6]}$$

Solution : Given, $A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}$

$$|A| = 2(120-45) - 3(-80-30) + 10(36+36)$$
$$= 2 \times 75 - 3 \times (-110) + 10 \times 72$$
$$= 150 + 330 + 720 = 1200 \neq 0$$

So, A is invertible.

$\therefore$ A^{-1} exists.

Let A_{ij} be the cofactors of elements a_{ij} in $A = [a_{ij}]$. Then,

$$A_{11} = (-1)^{1+1}\begin{bmatrix} -6 & 5 \\ 9 & -20 \end{bmatrix} = 120 - 45 = 75$$

$$A_{12} = (-1)^{1+2}\begin{bmatrix} 4 & 5 \\ 6 & -20 \end{bmatrix} = -(-80-30) = 110$$

$$A_{13} = (-1)^{1+3}\begin{bmatrix} 4 & -6 \\ 6 & 9 \end{bmatrix} = 36 + 36 = 72$$

$$A_{21} = (-1)^{2+1}\begin{bmatrix} 3 & 10 \\ 9 & -20 \end{bmatrix} = -(-60-90) = 150$$

$$A_{22} = (-1)^{2+2}\begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix} = -40 - 60 = -100$$

$$A_{23} = (-1)^{2+3}\begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} = -(18-18) = 0$$

$$A_{31} = (-1)^{3+1}\begin{bmatrix} 3 & 10 \\ -6 & 5 \end{bmatrix} = 15 + 60 = 75$$

$$A_{32} = (-1)^{3+2}\begin{bmatrix} 2 & 10 \\ 4 & 5 \end{bmatrix} = -(10-40) = 30$$

$$A_{33} = (-1)^{3-3}\begin{bmatrix} 2 & 3 \\ 4 & -6 \end{bmatrix} = -12 - 12 = -24$$

Cofactor matrix of $A = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}$

$$\text{adj } A = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}$$

Given system of equations is,

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=2$$
$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=5$$
$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=-4$$

Let $\frac{1}{x}=u,\ \frac{1}{y}=v,\ \frac{1}{z}=w$

$\therefore$ The given linear equation becomes

$$2u + 3v + 10w = 2$$
$$4u - 6v + 5w = 5$$
$$6u + 9v - 20w = -4$$

$$A = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}, \quad X = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$X = A^{-1}B$

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \\ -4 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 150+750-300 \\ 220-500-120 \\ 144+0+96 \end{bmatrix}$$

$$= \frac{1}{1200}\begin{bmatrix} 600 \\ -400 \\ 240 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/3 \\ 1/5 \end{bmatrix}$$

$u = 1/2 \Rightarrow x = 2$

$v = -1/3 \Rightarrow y = -3$

and $w = 1/5 \Rightarrow z = 5$

$\Rightarrow$ $x = 2,\ y = -3$ and $z = 5$ **Ans.**

●●

Mathematics 2017 (Delhi) — SET III

Time allowed : 3 hours | **Maximum marks : 100**

> **Note :Except for the following questions, all the remaining questions have been asked in previous sets.**

SECTION — B

12. If $y = \sin^{-1}\left(6x\sqrt{1-9x^2}\right)$, $\frac{-1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$, then find $\frac{dy}{dx}$. **[2]**

Solution : Given, $y = \sin^{-1}\left(6x\sqrt{1-9x^2}\right)$

Put $x = \frac{\sin\theta}{3} \Rightarrow \theta = \sin^{-1}3x.$

$$y = \sin^{-1}\left(6.\frac{\sin\theta}{3}\sqrt{1-\frac{9\sin^2\theta}{9}}\right)$$

$$\Rightarrow \quad y = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$\Rightarrow \quad y = \sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$$

$$\Rightarrow \quad y = \sin^{-1}(2\sin\theta\cos\theta)$$

$$\Rightarrow \quad y = \sin^{-1}(\sin 2\theta) = 2\theta = 2\sin^{-1}3x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{\sqrt{1-9x^2}} \times 3 = \frac{6}{\sqrt{1-9x^2}}$$ **Ans.**

SECTION — C

20. Solve the following L.P.P. graphically :

Maximise Z = 20x + 10y

Subject to the following constraints :

$$x + 2y \le 28,$$
$$3x + y \le 24,$$
$$x \ge 2$$
$$x, y \ge 0$$ **[4]**

Solution : We have

Maximise Z = 20x + 10y

Subject to the constraints :

$$x + 2y \le 28$$
$$3x + y \le 24$$
$$x \ge 2$$
$$x, y \ge 0$$

Converting the given inequalities into equations, we obtain the following equations :

$$x + 2y = 28$$
$$3x + y = 24$$
$$x = 2$$

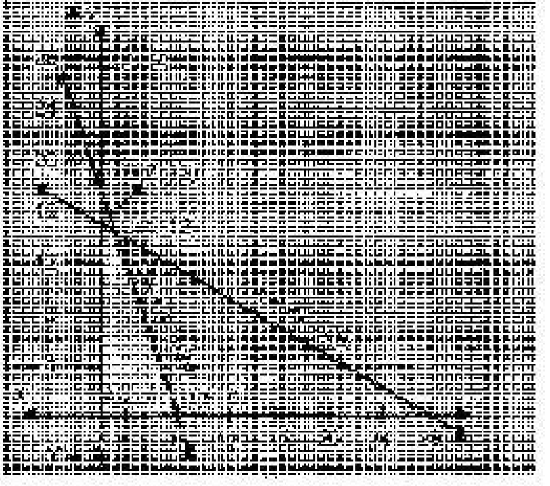
Then $x + 2y = 28$

x	0	28
y	14	0

$3x + y = 24$

x	0	8
y	24	0

$x = 2$ is a line which is parallel to y-axis.



Plotting these points on the graph, we get the shaded feasible region *i.e.*, ABCD

Corner point	Value of Z = 20x + 10y
A (2, 0)	Z = 40 + 0 = 40
B (2, 13)	Z = 40 + 130 = 170
C (4, 12)	Z = 80 + 120 = 200 ← Maximum
D (8, 0)	Z = 160 + 0 = 160

∴ Maximum value of Z is 200 at (4, 12). **Ans.**

21. Show that the family of curves for which $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$, is given by $x^2 - y^2 = cx$. **[4]**

Solution : Given family of curve,

$$\frac{dy}{dx} = \frac{x^2+y^2}{2xy} \quad ...(i)$$

Put $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

From equation (i),

$$v + x\frac{dv}{dx} = \frac{x^2+(vx)^2}{2x(vx)} = \frac{x^2(1+v^2)}{x^2(2v)}$$

$$\Rightarrow \quad x\frac{dv}{dx} = \frac{1+v^2}{2v} - v$$

$$= \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

$$\Rightarrow \frac{2v}{1-v^2}dv = \frac{dx}{x}$$

$$\Rightarrow \int \frac{2v}{1-v^2}dv = \int \frac{dx}{x}$$

$$\Rightarrow -\log|1-v^2| + \log C = \log x,$$

where log C is constant of integration.

$$\log\left|\frac{c}{1-\frac{y^2}{x^2}}\right| = \log x$$

$$\Rightarrow \log\left|\frac{cx^2}{x^2-y^2}\right| = \log x$$

$$\Rightarrow \frac{cx^2}{x^2-y^2} = x$$

$$\Rightarrow cx = x^2 - y^2$$ **Hence Proved.**

22. Find : $\int \frac{(3\sin x - 2)\cos x}{13-\cos^2 x - 7\sin x}dx$ **[4]**

Solution : Let $I = \int \frac{(3\sin x - 2)\cos x}{13-\cos^2 x - 7\sin x}dx$

$$= \int \frac{(3\sin x - 2)\cos x}{\sin^2 x - 7\sin x + 12}dx$$

$$(\because \cos^2 x = 1 - \sin^2 x)$$

Put $\sin x = t$

$\Rightarrow \cos x\, dx = dt$

$$I = \int \frac{(3t-2)}{t^2-7t+12}dt = \int \frac{3t-2}{(t-3)(t-4)}dt$$

Let, $\frac{3t-2}{(t-3)(t-4)} = \frac{A}{t-3} + \frac{B}{t-4}$ ...(A)

$$\Rightarrow \frac{3t-2}{(t-3)(t-4)} = \frac{A(t-4)+B(t-3)}{(t-3)(t-4)}$$

$$\Rightarrow 3t-2 = A(t-4) + B(t-3)$$

Equating coefficient of t and constant term on both sides, we get

$A + B = 3$ ...(i)

$-4A - 3B = -2$ ...(ii)

Solving the above two equations, we get

$$A = -7, B = 10$$

From equation (A),

$$\frac{3t-2}{(t-3)(t-4)} = \frac{-7}{t-3} + \frac{10}{t-4}$$

$$\int \frac{3t-2}{(t-3)(t-4)}dt = -7\int\frac{1}{t-3}dt + 10\int\frac{1}{t-4}dt$$

$$I = -7\log|t-3| + 10\log|t-4| + C,$$

where C is constant of integration.

$$I = 10\log|\sin x - 4| - 7\log|\sin x - 3| + C.$$

Ans.

23. Solve the following equation for x :

$\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$ **[4]**

Solution : Given equation is,

$$\cos(\tan^{-1}x) = \sin\left(\cot^{-1}\frac{3}{4}\right) \quad ...(i)$$

$$\Rightarrow \cos(\tan^{-1}x) = \cos\left[\frac{\pi}{2} - \cot^{-1}\frac{3}{4}\right]$$

$$\Rightarrow \tan^{-1}x = \frac{\pi}{2} - \cot^{-1}\frac{3}{4}$$

$$\Rightarrow \frac{\pi}{2} - \cot^{-1}x = \frac{\pi}{2} - \cot^{-1}\frac{3}{4}$$

$$\left(\because \tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}\right)$$

$$\Rightarrow \cot^{-1}x = \cot^{-1}\frac{3}{4}$$

or $x = \frac{3}{4}$ **Ans.**

SECTION — D

28. If A = $\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & -2 \\ -3 & 1 & -1 \end{bmatrix}$**, find A^{-1} and hence solve the system of equations $2x + y - 3z = 13$, $3x + 2y + z = 4$, $x + 2y - z = 8$** **[6]**

Solution : Given,

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ -3 & 1 & -1 \end{bmatrix}$$

$$|A| = 2(-2-2) - 3(-1+6) + 1(1+6)$$

$$= -8 - 15 + 7 = -16 \neq 0$$

So A is invertible.

$\therefore A^{-1}$ exists

Let A_{ij} be the cofactors of elements a_{ij} in $A = [a_{ij}]$. Then,

$$A_{11} = (-1)^{1+1}\begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix} = -2 - 2 = -4$$

$$A_{12} = (-1)^{1+2}\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} = -(-1+6) = -5$$

$$A_{13} = (-1)^{1+3}\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix} = 1 + 6 = 7$$

$$A_{21} = (-1)^{2+1}\begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} = -(-3-1) = 4$$

$$A_{22} = (-1)^{2+2}\begin{bmatrix} 2 & 1 \\ -3 & -1 \end{bmatrix} = -2+3 = 1$$

$$A_{23} = (-1)^{2+3}\begin{bmatrix} 2 & 3 \\ -3 & 1 \end{bmatrix} = -(2+9) = -11$$

$$A_{31} = (-1)^{3+1}\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix} = 6-2 = 4$$

$$A_{32} = (-1)^{3+2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = -(4-1) = -3$$

$$A_{33} = (-1)^{3+3}\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = 4-3 = 1$$

$$\therefore a_{ij} = \begin{bmatrix} -4 & -5 & 7 \\ 4 & 1 & -11 \\ 4 & -3 & 1 \end{bmatrix}$$

$$\text{adj. } A = [A_{ij}]^T = \begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj. } A = \frac{-1}{16}\begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix}$$

From the given linear equations, we have

Let $$C = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 13 \\ 4 \\ 8 \end{bmatrix}$$

by matrix method,

$$X = C^{-1}D = (A^T)^{-1}D = (A^{-1})^TD$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{16}\begin{bmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{bmatrix}^T \begin{bmatrix} 13 \\ 4 \\ 8 \end{bmatrix}$$

$$= \frac{-1}{16}\begin{bmatrix} -4 & -5 & 7 \\ 4 & 1 & -11 \\ 4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 13 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{16}\begin{bmatrix} -52-20+56 \\ 52+4-88 \\ 52-12+8 \end{bmatrix}$$

$$= \frac{-1}{16}\begin{bmatrix} -16 \\ -32 \\ 48 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow$ $x = 1, y = 2, z = -3.$ **Ans.**

29. Find the particular solution of the differential equation. $\tan x \cdot \dfrac{dy}{dx} = 2x\tan x + x^2 - y$; ($\tan x \neq 0$) given that $y = 0$ when $x = \dfrac{\pi}{2}$. **[6]**

Solution : Given differential equation is,

$$\tan x.\frac{dy}{dx} = 2x\tan x + x^2 - y$$

$$\Rightarrow \frac{dy}{dx} = 2x + x^2\cot x - y\cot x$$

$$\Rightarrow \frac{dy}{dx} + y\cot x = 2x + x^2\cot x,$$

which is of the form

$$\frac{dy}{dx} + Py = Q,$$

where $P = \cot x$, $Q = 2x + x^2\cot x$

$$\text{I. F.} = e^{\int P dx} = e^{\int \cot x\, dx} = e^{\log|\sin x|} = \sin x$$

$\therefore$ Required solution is

$$y.\sin x = \int (2x + x^2\cot x)\sin x\, dx + C$$
$$= \int 2x\sin x\, dx + \int x^2\cot x\sin x\, dx + C$$
$$= 2\int x\sin x\, dx + \int x^2\cos x\, dx + C$$
$$= 2\left[x.(-\cos x) - \int(-\cos x)dx\right] + x^2.\sin x - \int 2x.\sin x\, dx + C$$
$$= -2x\cos x + 2\sin x + x^2\sin x - 2\left[x.(-\cos x) - \int -\cos x\, dx\right] + C$$
$$= -2x\cos x + 2\sin x + x^2\sin x + 2x\cos x - 2\sin x + C$$

$$\therefore \quad y\sin x = x^2\sin x + C \qquad \text{...(i)}$$

Give that $y = 0$ when $x = \dfrac{\pi}{2}$

$$0 = \frac{\pi^2}{4}\sin\frac{\pi}{2} + C$$

$$\Rightarrow C = -\frac{\pi^2}{4}$$

From equation (i)

$$y\sin x = x^2\sin x - \frac{\pi^2}{4}$$

$$\Rightarrow (x^2 - y)\sin x = \frac{\pi^2}{4}$$ **Ans.**

●●