Mathematics 2014 (Outside Delhi) SET I

Time allowed : 3 hours Maximum marks : 100

SECTION — A

1. **If R = [(x, y) : x + 2y = 8] is a relation on N, write the range of R.** [1]

Solution : The given relation on N is

$$R = [(x, y) : x + 2y = 8]$$

Since both $x, y \in N$

$\therefore$ x can take values 2, 4, 6 for other values of $y \in N$.

For $x = 2$, $2 + 2y = 8$

$\Rightarrow$ $y = 3$

For $x = 4$, $4 + 2y = 8$

$\Rightarrow$ $y = 2$

For $x = 6$, $6 + 2y = 8$

$\Rightarrow$ $y = 1$

$\therefore$ $R = \{(2, 3), (4, 2), (6, 1)\}$

$\therefore$ The range of R = Set of second element's

$= \{1, 2, 3\}$. **Ans.**

2. **If $\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}, xy < 1$, then write the value of $x + y + xy$.** [1]

Solution : Given,

$$\tan^{-1} x + \tan^{-1} y = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \frac{x+y}{1-xy} = \frac{\pi}{4}$$

$$\Rightarrow \frac{x+y}{1-xy} = \tan\frac{\pi}{4}$$

$$\Rightarrow \frac{x+y}{1-xy} = 1$$

$$\Rightarrow x + y = 1 - xy$$

$\therefore$ $x + y + xy = 1$. **Ans.**

3. **If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix.** [1]

Solution :

$7A - (I + A)^3 = 7A - (I^3 + A^3 + 3I^2A + 3IA^2)$

$[\because (a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2]$

$= 7A - (I + A^2.A + 3IA + 3IA^2)$

$[\because I^n = I \ \forall n \in N]$

$= 7A - (I + A^2 + 3A + 3IA)$

$[\because A^2 = A, IA = A]$

$= 7A - (I + A + 3A + 3A)$

$= 7A - I - 7A = -I$. **Ans.**

4. **If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of $x + y$.** [1]

Solution : Given,

$$\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$$

Comparing the corresponding elements, we get

$$x - y = -1, z = 4$$

$$2x - y = 0, w = 5$$

Solving these equations, we get

$$x = 1, y = 2$$

$\therefore$ $x + y = 1 + 2 = 3$. **Ans.**

5. **If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x.** [1]

Solution : Given,

$$\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$$

$\Rightarrow$ $3x \times 4 - (-2) \times 7 = 8 \times 4 - 6 \times 7$

$\Rightarrow$ $12x + 14 = 32 - 42$

$\Rightarrow$ $12x = -10 - 14 = -24$

$\therefore$ $x = -2$. **Ans.**

6. **If $f(x) = \int_0^x t \sin t \, dt$, then write the value of $f'(x)$.** [1]

Solution : Given, $f(x) = \int_0^x t \sin t \, dt$,

Integrating by parts, we get

$$\int_0^x t \sin t \, dt = t(-\cos t)\Big]_0^x - \int_0^x 1.(-\cos t)dt$$

$$f(x) = \left[-t\cos t + \sin t\right]_0^x$$

$$= -x\cos x + \sin x$$

$\therefore$ $f(x) = -x\cos x + \sin x$

Differentiating w.r.t. x, we get

$$f'(x) = -[1.\cos x - x\sin x] + \cos x$$

$$= -\cos x + x\sin x + \cos x$$

$= x\sin x$ **Ans.**

7. **Evaluate : $\int_2^4 \frac{x}{x^2+1} dx$.** [1]

Solution : Let $I = \int_2^4 \frac{x}{x^2+1} dx$

Putting $x^2 + 1 = t$

$\Rightarrow$ $2x \, dx = dt$

$\Rightarrow$ $x \, dx = \frac{1}{2} dt$

Also, $x = 2 \Rightarrow t = 5$

and $x = 4 \Rightarrow t = 17$

$$\therefore \quad I = \frac{1}{2}\int_5^{17}\frac{dt}{t} = \frac{1}{2}[\log t]_5^{17}$$

$$= \frac{1}{2}[\log 17 - \log 5]$$

$$= \frac{1}{2}\log\left(\frac{17}{5}\right). \quad \textbf{Ans.}$$

8. Find the value of 'p' for which the vectors $3\hat{i} + 2\hat{j} + 9\hat{k}$ and $\hat{i} - 2p\hat{j} + 3\hat{k}$ are parallel. **[1]**

Solution : Let $\vec{a} = 3\hat{i} + 2\hat{j} + 9\hat{k}$ ...(i)

and $\vec{b} = \hat{i} - 2p\hat{j} + 3\hat{k}$ ...(ii)

Since $\vec{a}$ and $\vec{b}$ are parallel,

$$\therefore \quad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$\Rightarrow \quad \frac{3}{1} = \frac{2}{-2p} = \frac{9}{3}$$

$$\Rightarrow \quad \frac{3}{1} = \frac{2}{-2p}$$

$$\Rightarrow \quad 3 = \frac{1}{-p}$$

$$\Rightarrow \quad p = \frac{-1}{3} \quad \textbf{Ans.}$$

9. Find $\vec{a}.(\vec{b} \times \vec{c})$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$. **[1]**

Solution : Given,

$$\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$$

$$\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$$

$$(\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= \hat{i}(4-1) - \hat{j}(-2-3) + \hat{k}(-1-6)$$

$$= 3\hat{i} + 5\hat{j} - 7\hat{k}$$

$$\therefore \quad \vec{a}.(\vec{b} \times \vec{c}) = (2\hat{i} + \hat{j} + 3\hat{k}).(3\hat{i} + 5\hat{j} - 7\hat{k})$$

$$= 2 \times 3 + 1 \times 5 + 3 \times (-7)$$

$$= 6 + 5 - 21 = -10. \quad \textbf{Ans.}$$

10. If the Cartesian equations of a line are $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$, write the vector equation for the line. **[1]**

Solution : The Cartesian equations of a line are

$$\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4} \quad ...(i)$$

$$\Rightarrow \quad \frac{x-3}{-5} = \frac{y-(-4)}{7} = \frac{z-3}{2} = \lambda(say)$$

$$\Rightarrow \quad x = 3 - 5\lambda,$$

$$y = -4 + 7\lambda,$$

$$z = 3 + 2\lambda$$

Now, $\vec{a} = 3\hat{i} - 4\hat{j} + 3\hat{k}$

and $\vec{b} = -5\hat{i} + 7\hat{j} + 2\hat{k}$.

$\therefore$ The vector equation of the line (i) is

$$\vec{r} = \vec{a} + \lambda\vec{b}$$

$$\Rightarrow \vec{r} = (3\hat{i} - 4\hat{j} + 3\hat{k}) + \lambda(-5\hat{i} + 7\hat{j} + 2\hat{k}). \quad \textbf{Ans.}$$

SECTION — B

11. If the function $f : R \rightarrow R$ be given by $f(x) = x^2 + 2$ and $g : R \rightarrow R$ be given by $g(x) = \frac{x}{x-1}, x \neq 1$, find fog and gof and hence find fog (2) and gof (–3). **[4]**

Solution : Given, $f : R \rightarrow R$

such that, $f(x) = x^2 + 2$ ...(i)

and $g : R \rightarrow R$

such that, $g(x) = \frac{x}{x-1}, x \neq 1$ ...(ii)

Now, $fog : R \rightarrow R$

such that,
$$(fog)(x) = f(g(x)) = f\left(\frac{x}{x-1}\right)$$

$$= \left(\frac{x}{x-1}\right)^2 + 2 \ (x \neq 1)$$

$$\therefore \quad (fog)(2) = \left(\frac{2}{2-1}\right)^2 + 2 = 6$$

Also, $gof : R \rightarrow R$

Such that, $(gof)(x) = g(f(x))$

$$= g(x^2 + 2) = \frac{x^2 + 2}{x^2 + 2 - 1}$$

$$= \frac{x^2 + 2}{x^2 + 1}$$

$$\therefore \quad (gof)(-3) = \frac{(-3)^2 + 2}{(-3)^2 + 1} = \frac{11}{10}. \quad \textbf{Ans.}$$

12. Prove that :

$$\tan^{-1}\left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x, \frac{-1}{\sqrt{2}} \leq x \leq 1.$$ **[4]**

Solution : L. H. S.

$$= \tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]\left(\frac{-1}{\sqrt{2}}\le x\le 1\right)$$

Putting

$x = \cos\theta$, we get

$$= \tan^{-1}\left[\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right],$$

$$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}\right]\left(0\le x\le\frac{3\pi}{4}\right)$$

$$\left(\because \cos\theta = 2\cos^2\frac{\theta}{2}-1 = 1-2\sin^2\frac{\theta}{2}\right)$$

$$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2}\left(1-\tan\frac{\theta}{2}\right)}{\sqrt{2}\cos\frac{\theta}{2}\left(1+\tan\frac{\theta}{2}\right)}\right]$$

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right] = \frac{\pi}{4}-\frac{1}{2}\theta$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x = \text{R.H.S.}$$ **Hence Proved.**

OR

If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$, find the value of x.

Solution : Given,

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \tan^{-1}\left[\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\frac{x-2}{x-4}\cdot\frac{x+2}{x+4}}\right] = \frac{\pi}{4}$$

$$\left(\because \tan^{-1}x+\tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right)$$

$$\Rightarrow \quad \frac{(x-2)(x+4)+(x+2)(x-4)}{(x^2-16)-(x^2-4)} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{x^2+2x-8+x^2-2x-8}{-12} = 1$$

$$\Rightarrow \quad 2x^2-16 = -12$$

$$\Rightarrow \quad 2x^2 = 4$$

$$\Rightarrow \quad x^2 = 2$$

$$\therefore \quad x = \pm\sqrt{2}\text{. Ans.}$$

13. Using properties of determinants, prove that:

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3$$ **[4]**

Solution : Taking L.H.S.

Let $\quad \Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$

Applying $R_3 \to R_3 - 2R_2$, we get

$$\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 0 & 0 & -x \end{vmatrix}$$

Taking x common from C_2 and C_3, we get

$$\Delta = x^2\begin{vmatrix} x+y & 1 & 1 \\ 5x+4y & 4 & 2 \\ 0 & 0 & -1 \end{vmatrix}$$

Expanding along R_3, we get

$$\Delta = x^2(-1)\begin{vmatrix} x+y & 1 \\ 5x+4y & 4 \end{vmatrix}$$

$$= -x^2[4(x+y)-(5x+4y)]$$

$$= -x^2(4x+4y-5x-4y)$$

$$= -x^2.(-x) = x^3 = \text{R. H. S.}$$

Hence Proved.

14. Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$ if $x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$. **[4]**

Solution : Given,

$$x = ae^{\theta}(\sin\theta - \cos\theta) \quad \text{...(i)}$$

Differentiating w. r. t. θ, we get

$$\frac{dx}{d\theta} = a[e^{\theta}(\sin\theta-\cos\theta)+e^{\theta}(\cos\theta+\sin\theta)]$$

$$= 2ae^{\theta}\sin\theta$$

and $\quad y = ae^{\theta}(\sin\theta+\cos\theta) \quad$ ...(ii)

Differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = a[e^{\theta}(\sin\theta+\cos\theta)+e^{\theta}(\cos\theta-\sin\theta)]$$

$$= 2ae^{\theta}\cos\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta}$$

$$\therefore \quad \frac{dy}{dx} = \cot\theta$$

$$\text{At } \theta = \frac{\pi}{4}; \frac{dy}{dx} = \cot\frac{\pi}{4} = 1. \quad \textbf{Ans.}$$

15. If $y = Pe^{ax} + Qe^{bx}$, Show that

$$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0. \quad [4]$$

Solution : Given,

$$y = Pe^{ax} + Qe^{bx} \quad ...(i)$$

Differentiating w. r. t. x, we get

$$\frac{dy}{dx} = P.e^{ax}.a + Q.e^{bx}.b \quad ...(ii)$$

Again differentiating, we get

$$\frac{d^2y}{dx^2} = aPe^{ax}.a + bQe^{bx}.b$$
$$= a^2Pe^{ax} + b^2Qe^{bx}$$

$$\therefore \text{ L. H. S.} = \frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby$$
$$= a^2Pe^{ax} + b^2Qe^{bx} - (a+b)(aPe^{ax} + bQe^{bx}) + ab(Pe^{ax} + Qe^{bx}) \quad ...(iii)$$

Using (i), (ii) and (iii),

$$= e^{ax}[a^2P - (a+b)aP + abP]$$
$$+ e^{bx}[b^2Q - (a+b)bQ + abQ]$$
$$= e^{ax}.0 + e^{bx}.0$$
$$= 0 = \text{R. H. S.} \quad \textbf{Hence Proved.}$$

16. Find the value (s) of x for which $y = [x(x-2)]^2$ is an increasing function. **[4]**

Solution : Given

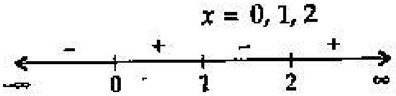
$$y = [x(x-2)]^2$$
$$\Rightarrow \quad y = x^2(x-2)^2$$
$$= f(x) \text{ (Let)}$$

Differentiating w.r.t. x, we get

$$f'(x) = \frac{dy}{dx}$$
$$= 2x(x-2)^2 + 2x^2(x-2)$$
$$= 2x(x-2)(x-2+x)$$
$$= 4x(x-1)(x-2).$$

For y to be an increasing function, $\frac{dy}{dx} > 0$

$$\Rightarrow \quad x(x-1)(x-2) > 0$$
$$\Rightarrow \quad x = 0, 1, 2$$



Interval	Test Value	Sign of $f'(x)$ $f'(x) = 4x(x-1)(x-2)$	Nature of function $f(x) = y$
$(-\infty, 0)$	$x = -1$	$(-)(-)(-) = - < 0$	Strictly decreasing
$(0, 1)$	$x = 0.5$	$(+)(-)(-) = + > 0$	Strictly increasing
$(1, 2)$	$x = 1.5$	$(+)(+)(-) = - < 0$	Strictly decreasing
$(2, \infty)$	$x = 3$	$(+)(+)(+) = + > 0$	Strictly increasing

$\therefore$ y is an increasing function in $[0, 1] \cup [2, \infty)$ **Ans.**

OR

Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(\sqrt{2}a, b)$.

Solution : The given curve is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Differentiating w.r. t. x, we get

$$\frac{2x}{a^2} - \frac{2y}{b^2}.\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\frac{x}{a^2}}{\frac{y}{b^2}} = \frac{b^2}{a^2}.\frac{x}{y}$$

At $P(\sqrt{2}a, b)$

Slope of the tangent,

$$m_1 = \frac{b^2}{a^2}.\frac{\sqrt{2}a}{b} = \sqrt{2}.\frac{b}{a}$$

and slope of the normal,

$$m_2 = -\frac{a}{b\sqrt{2}}$$

$\therefore$ Equation of the tangent at P is

$$y - b = \sqrt{2}\frac{b}{a}(x - \sqrt{2}a)$$
$$\Rightarrow \quad y = \sqrt{2}\frac{b}{a}x - 2b + b$$
$$\Rightarrow \quad y = \sqrt{2}\frac{b}{a}x - b$$
$$\Rightarrow \quad \sqrt{2}bx - ay - ab = 0$$

Also equation of the normal at P is

$$y - b = \frac{-a}{\sqrt{2}b}(x - \sqrt{2}a)$$

$$\Rightarrow \quad y = \frac{-ax}{\sqrt{2}b} + b + \frac{a^2}{b}.$$

$$\Rightarrow \quad \sqrt{2}\,by - \sqrt{2}b^2 = -ax + \sqrt{2}a^2$$

$$\Rightarrow \quad ax + \sqrt{2}\,by - \sqrt{2}(a^2 + b^2) = 0 \qquad \textbf{Ans.}$$

17. Evaluate : $\int_0^{\pi} \frac{4x \sin x}{1+\cos^2 x}dx.$ **[4]**

Solution : Let, $\quad I = \int_0^{\pi} \frac{4x\sin x}{1+\cos^2 x}dx \quad$ ...(i)

$$\Rightarrow \quad I = \int_0^{\pi} \frac{4(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\therefore \quad I = \int_0^{\pi} \frac{4(\pi - x)\sin x}{1+\cos^2 x}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow \quad I = 2\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$$

Putting, $\quad \cos x = t$

$\Rightarrow \quad \sin x\, dx = -dt$

Also, $\quad x = 0 \Rightarrow t = 1$

and $\quad x = \pi \Rightarrow t = -1$

$$\therefore \quad I = 2\pi \int_1^{-1} \frac{-dt}{1+t^2}$$

$$= 2\pi \int_{-1}^{1} \frac{dt}{t^2+1}$$

$$\left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

$$= 2\pi\left[\tan^{-1} t\right]_{-1}^{1}$$

$$= 2\pi[\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$= 2\pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$= 2\pi\left[\frac{2\pi}{4}\right] = \pi^2 \qquad \textbf{Ans.}$$

OR

Evaluate : $\int \frac{x+2}{\sqrt{x^2+5x+6}}dx.$

Solution : Let, $\quad x + 2 = A\frac{d}{dx}(x^2 + 5x + 6) + B$

$\Rightarrow \quad x + 2 = A(2x + 5) + B \quad$ ...(i)

$\Rightarrow \quad x + 2 = 2Ax + 5A + B$

On equating the coefficients of x and constant term on both sides, we get

$$2A = 1$$

$$\Rightarrow \quad A = \frac{1}{2}$$

and $\quad 5A + B = 2$

$$\Rightarrow \quad \frac{5}{2} + B = 2$$

$$\Rightarrow \quad B = 2 - \frac{5}{2} = -\frac{1}{2}$$

$\therefore$ Equation (i) becomes,

$$x + 2 = \frac{1}{2}(2x+5) - \frac{1}{2}$$

$$\therefore \quad \int \frac{x+2}{\sqrt{x^2+5x+6}}dx = \int \frac{\frac{1}{2}(2x+5) - \frac{1}{2}}{\sqrt{x^2+5x+6}}dx$$

$$= \frac{1}{2}\int \frac{2x+5}{\sqrt{x^2+5x+6}}dx - \frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}$$

Putting, $x^2 + 5x + 6 = t$

$\Rightarrow \quad (2x + 5)\,dx = dt$

$$= \frac{1}{2}\int t^{\frac{-1}{2}}dt - \frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}} + C$$

$$= \frac{1}{2}\frac{t^{\frac{1}{2}}}{\frac{1}{2}} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{\left(x+\frac{5}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

$$\left[\because \int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right|\right]$$

$$= \sqrt{x^2+5x+6} - \frac{1}{2}\log\left|\left(x+\frac{5}{2}\right) + \sqrt{x^2+5x+6}\right| + C$$

Ans.

18. Find the particular solution of the differential equation $\frac{dy}{dx} = 1 + x + y + xy$, given that $y = 0$ when $x = 1$. **[4]**

Solution : The given differential equation is

$$\frac{dy}{dx} = 1 + x + y + xy$$

$$\Rightarrow \quad \frac{dy}{dx} = (1 + x) + (1 + x)y$$

$$= (1 + x)(1 + y)$$

$$\Rightarrow \quad \frac{dy}{1+y} = (1 + x)dx$$

Integrating both sides, we get

$$\int \frac{dy}{1+y} = \int (1+x)dx + C$$

$$\Rightarrow \quad \log(1+y) = x+\frac{x^2}{2}+C$$

Putting, $y = 0$,

when $x = 1$, we get

$$\log 1 = 1+\frac{1}{2}+C$$

$$\Rightarrow \quad 0 = \frac{3}{2}+C$$

$$\Rightarrow \quad C = -\frac{3}{2}$$

$\therefore$ The particular solution of the given differential equation is

$$\log(1+y) = x+\frac{x^2}{2}-\frac{3}{2}. \quad \textbf{Ans.}$$

19. Solve the differential equation $(1+x^2)\frac{dy}{dx}+y = e^{\tan^{-1}x}$. **[4]**

Solution : The given differential equation is

$$(1+x^2)\frac{dy}{dx}+y = e^{\tan^{-1}x} \quad ...(i)$$

Rewriting the given differential equation, we get

$$\frac{dy}{dx}+\frac{1}{1+x^2}.y = \frac{e^{\tan^{-1}x}}{1+x^2},$$

which is a linear differential equation of the form

$$\frac{dy}{dx}+Py = Q$$

Here, $$P = \frac{1}{1+x^2}$$

and $$Q = \frac{e^{\tan^{-1}x}}{1+x^2}$$

$$\therefore \quad \int P\,dx = \int\frac{dx}{1+x^2} = \tan^{-1}x$$

Now, $$\text{I. F.} = e^{\int P\,dx} = e^{\tan^{-1}x}$$

Hence the solution is

$$y(\text{I.F.}) = \int Q.(\text{I.F.})dx+C$$

$$y.e^{\tan^{-1}x} = \int\frac{e^{\tan^{-1}x}}{1+x^2}.e^{\tan^{-1}x}dx+C$$

$$= \int\frac{\left(e^{\tan^{-1}x}\right)^2 dx}{1+x^2}+C$$

Putting, $\tan^{-1}x = t$

$$\Rightarrow \quad \frac{1}{1+x^2}dx = dt$$

$$\Rightarrow \quad ye^{\tan^{-1}x} = \int e^{2t}dt+C$$

$$= \frac{1}{2}e^{2t}+C$$

$$y.e^{\tan^{-1}x} = \frac{1}{2}e^{2\tan^{-1}x}+C.$$

$$\therefore \quad y = \frac{1}{2}e^{\tan^{-1}x}+Ce^{-\tan^{-1}x} \quad \textbf{Ans.}$$

20. Show that the four points A, B, C and D with position vectors $4\hat{i}+5\hat{j}+\hat{k}$, $-\hat{j}-\hat{k}$, $3\hat{i}+9\hat{j}+4\hat{k}$ and $4(-\hat{i}+\hat{j}+\hat{k})$ respectively are coplanar. **[4]**

Solution : Given the position vectors are

$$A(4\hat{i}+5\hat{j}+\hat{k});\ B(-\hat{j}-\hat{k});\ C(3\hat{i}+9\hat{j}+4\hat{k});$$

and $D\left[4(-\hat{i}+\hat{j}+\hat{k})\right]$.

These points will be coplanar if

$$\left[\vec{AB}, \vec{AC}, \vec{AD}\right] = 0$$

Now, $$\vec{AB} = (-\hat{j}-\hat{k})-(4\hat{i}+5\hat{j}+\hat{k})$$

$$= -4\hat{i}-6\hat{j}-2\hat{k}$$

$$\vec{AC} = (3\hat{i}+9\hat{j}+4\hat{k})-(4\hat{i}+5\hat{j}+\hat{k})$$

$$= -\hat{i}+4\hat{j}+3\hat{k}$$

$$\vec{AD} = (-4\hat{i}+4\hat{j}+4\hat{k})-(4\hat{i}+5\hat{j}+\hat{k})$$

$$= -8\hat{i}-\hat{j}+3\hat{k}.$$

$$\therefore \left[\vec{AB}, \vec{AC}, \vec{AD}\right] = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

$$= -4(12+3)+6(-3+24) -2(1+32)$$

$$= -60+126-66$$

$$= -126+126 = 0$$

$\therefore$ The given points A, B, C and D are coplanar.

Hence Proved

OR

The scalar product of the vector $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c} = \lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$.

Solution : Given, $\vec{a} = \hat{i}+\hat{j}+\hat{k}$;

$$\vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$$

and $$\vec{c} = \lambda\hat{i}+2\hat{j}+3\hat{k}$$

$$\Rightarrow \quad \vec{b}+\vec{c} = (2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$$

The unit vector along $\vec{b}+\vec{c}$ is

$$\vec{p} = \frac{\vec{b}+\vec{c}}{|\vec{b}+\vec{c}|}$$

$$\Rightarrow \quad \vec{p} = \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}}$$

$$\Rightarrow \quad = \frac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{\lambda^2+4\lambda+44}} \quad \text{...(i)}$$

Given, $\quad \vec{a}.\vec{p} = 1$

$$\Rightarrow \quad \frac{(\hat{i}+\hat{j}+\hat{k}).[(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}]}{\sqrt{\lambda^2+4\lambda+44}} = 1$$

$$\Rightarrow \quad \frac{(2+\lambda)+6-2}{\sqrt{\lambda^2+4\lambda+44}} = 1$$

$$\Rightarrow \quad \sqrt{\lambda^2+4\lambda+44} = \lambda+6$$

Squaring on both sides, we get

$$\lambda^2+4\lambda+44 = \lambda^2+12\lambda+36$$

$$\Rightarrow \quad 8\lambda = 8$$

$$\Rightarrow \quad \lambda = 1 \quad \text{...(ii)}$$

and the required unit vector is

$$\vec{p} = \frac{(2+1)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{1+4+44}}$$

[Using (i) and (ii)]

$$= \frac{1}{7}(3\hat{i}+6\hat{j}-2\hat{k}). \quad \textbf{Ans.}$$

21. A line passes through (2, –1, 3) and is perpendicular to the lines $\vec{r} = (\hat{i}+\hat{j}-\hat{k}) + \lambda(2\hat{i}-2\hat{j}+\hat{k})$ and $\vec{r} = (2\hat{i}-\hat{j}-3\hat{k}) + \mu(\hat{i}+2\hat{j}+2\hat{k})$. Obtain its equation in vector and Cartesian form. [4]

Solution : The given lines are

$$\vec{r} = (\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k}) \quad \text{...(i)}$$

$$\vec{r} = (2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k}) \quad \text{...(ii)}$$

Equation of any line passes through (2, –1, 3) with direction cosines l, m, n is

$$\vec{r} = (2\hat{i}-\hat{j}+3\hat{k})+\lambda(l\hat{i}+m\hat{j}+n\hat{k}) \quad \text{...(iii)}$$

Now line (i) and (ii) are perpendicular to (iii), we get

$$2l-2m+n = 0 \quad \text{...(iv)}$$

$$l+2m+2n = 0 \quad \text{...(v)}$$

Solving equations (iv) and (v), we get

$$\frac{l}{-4-2} = \frac{m}{1-4} = \frac{n}{4+2}$$

$$\Rightarrow \quad \frac{l}{2} = \frac{m}{1} = \frac{n}{-2}$$

$\therefore$ From (iii), the required line in vector form is

$$\vec{r} = (2\hat{i}-\hat{j}+3\hat{k})+\lambda(2\hat{i}+\hat{j}-2\hat{k})$$

Also cartesian Equation is

$$\frac{x-2}{2} = \frac{y+1}{1} = \frac{z-3}{-2} \quad \textbf{Ans.}$$

22. An experiment succeeds thrice as often as it fails. Find the probability that in the next five trials, there will be at least 3 successes. [4]

Solution : Let, p = Probability of success

$$= \frac{3}{3+1} = \frac{3}{4}$$

q = Probability of failure

$$= 1-\frac{3}{4} = \frac{1}{4}$$

Here, $\quad n = 5$

$\therefore$ Probability of at least 3 successes

$$= P(X \geq 3)$$

$$= P(X=3)+P(X=4)+P(X=5)$$

$$= {}^5C_3p^3q^2 + {}^5C_4p^4q^1 + {}^5C_5p^5q^0$$

$$= 10\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 + 5\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right) + \left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^0$$

$$= \frac{270+405+243}{4^5}$$

$$= \frac{918}{1024} = \frac{459}{512}. \quad \textbf{Ans.}$$

SECTION — C

23. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 1,600. School B wants to spend ₹ 2,300 to award its 4, 1 and 3 students on the

respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is ₹ 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award. **[6]**

Solution : Given the awards for sincerity, truthfulness and helpfulness are ₹ x, ₹ y and ₹ z respectively.

$$\therefore \quad 3x + 2y + z = 1{,}600$$
$$4x + y + 3z = 2{,}300$$
$$x + y + z = 900$$

The given equation can be written in matrix form,

$$AX = B \qquad ...(i)$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1{,}600 \\ 2{,}300 \\ 900 \end{bmatrix}$$

Here, $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$B = \begin{bmatrix} 1{,}600 \\ 2{,}300 \\ 900 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix}$

$$= 3(1-3) - 2(4-3) + 1(4-1)$$
$$= -6 - 2 + 3$$
$$= -5 \neq 0$$

$\Rightarrow A^{-1}$ exists.

For adj A,

$A_{11} = (1-3) = -2$, $A_{12} = -(4-3) = -1$; $A_{13} = (4-1) = 3$

$A_{21} = -(2-1) = -1$, $A_{22} = (3-1) = 2$, $A_{23} = -(3-2) = -1$

$A_{31} = (6-1) = 5$, $A_{32} = -(9-4) = -5$, $A_{33} = (3-8) = -5$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= -\frac{1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

From (i), $X = A^{-1} B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1{,}600 \\ 2{,}300 \\ 900 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -1{,}000 \\ -1{,}500 \\ -2{,}000 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$$

$\Rightarrow \quad x =$ ₹ 200,
$y =$ ₹ 300
and $z =$ ₹ 400

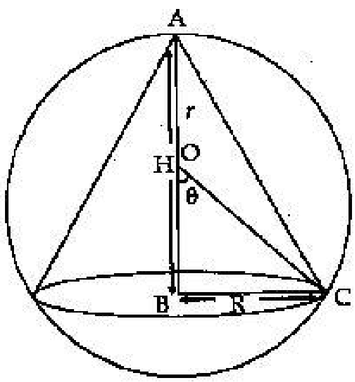
Apart from the three values, sincerity, truthfulness and helpfulness, another value for award should be discipline. **Ans.**

24. Show that the altitude of the right circular cone of maximum volume that can be described in a sphere of radius r is $\frac{4r}{3}$. Also show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere. **[6]**

Solution : From the figure OA = OC = r (Radius of the sphere)

From right angled Δ OBC,

$$BC = r \sin\theta,$$
$$OB = r \cos\theta$$



V = Volume of the inscribed cone

$$= \frac{1}{3}\pi R^2 H = \frac{1}{3}\pi (BC)^2 . AB$$

$$= \frac{1}{3}\pi r^2 \sin^2\theta (r + r\cos\theta) \qquad ...(i)$$

$$[\because AB = AO + OB]$$

$$\Rightarrow \quad V = \frac{\pi}{3} r^3 (\sin^2\theta + \sin^2\theta\cos\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{dV}{d\theta} = \frac{\pi r^3}{3}[2\sin\theta\cos\theta + 2\sin\theta\cos^2\theta - \sin^3\theta]$$

For maxima or minima,

$$\frac{dV}{d\theta} = 0$$

$$\Rightarrow \sin\theta[2\cos\theta + 2\cos^2\theta - \sin^2\theta] = 0$$

$$\Rightarrow \quad 2\cos\theta + 2\cos^2\theta - (1 - \cos^2\theta) = 0$$

$$[\because \sin\theta \neq 0]$$

$$\Rightarrow \quad 3\cos^2\theta + 2\cos\theta - 1 = 0$$

$$\Rightarrow \quad 3\cos^2\theta + 3\cos\theta - \cos\theta - 1 = 0$$

$$\Rightarrow 3\cos\theta(\cos\theta + 1) - 1(\cos\theta + 1) = 0$$

$$\Rightarrow \quad (3\cos\theta - 1).(\cos\theta + 1) = 0$$

$$\Rightarrow \quad 3\cos\theta - 1 = 0$$

$$[\because \cos\theta \neq 0]$$

$$\Rightarrow \quad \cos\theta = \frac{1}{3}$$

$$\therefore \quad H = r + r\cos\theta$$

$$= r + r.\frac{1}{3} = \frac{4r}{3}$$

Also V changes sign from + ve to – ve for this value of θ

⇒ V is maximum.

∴ Maximum volume of the inscribed cone,

$$V = \frac{\pi}{3} r^3 (1 - \cos^2\theta)(1 + \cos\theta) \text{ [From (i)]}$$

$$= \frac{\pi}{3} r^3 \left(1 - \frac{1}{9}\right)\left(1 + \frac{1}{3}\right)$$

$$= \frac{\pi}{3} r^3 \frac{32}{27} = \frac{8}{27}\left(\frac{4}{3}\pi r^3\right)$$

$$= \frac{8}{27} \text{ Volume of the sphere.}$$

Hence Proved.

25. Evaluate : $\int \frac{1}{\cos^4 x + \sin^4 x} dx.$ **[6]**

Solution : Let,

$$I = \int \frac{1}{\cos^4 x + \sin^4 x} dx$$

$$= \int \frac{\sec^4 x}{1 + \tan^4 x} dx$$

$$= \int \frac{\sec^2 x.\sec^2 x}{1 + \tan^4 x} dx$$

$$= \int \frac{(\tan^2 + 1)\sec^2 x}{1 + \tan^4 x} dx$$

$$(\because \sec^2 x = 1 + \tan^2 x)$$

Putting $\tan x = t$

$$\Rightarrow \quad \sec^2 x\, dx = dt$$

$$\therefore \quad I = \int \frac{t^2 + 1}{t^4 + 1} dt = \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2}} dt$$

$$= \int \frac{1 + \frac{1}{t^2}}{\left(t - \frac{1}{t}\right)^2 + 2} dt$$

Putting, $t - \frac{1}{t} = y$

$$\Rightarrow \quad \left(1 + \frac{1}{t^2}\right) dt = dy$$

$$\therefore \quad I = \int \frac{dy}{y^2 + (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + C$$

$$\left(\because \int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C\right)$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - \frac{1}{\tan x}}{\sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{2}}\right) + C$$

Ans.

26. Using integration, find the area of the region bounded by the triangle whose vertices are (– 1, 2), (1, 5) and (3, 4). **[6]**

Solution : Let A (– 1, 2); B (1, 5) and C (3, 4)

Equation of AB is

$$y - 5 = \frac{5 - 2}{1 + 1}(x - 1)$$

$$\Rightarrow \quad y = \frac{3}{2}x + \frac{7}{2} \quad \text{...(i)}$$

Equation of BC is

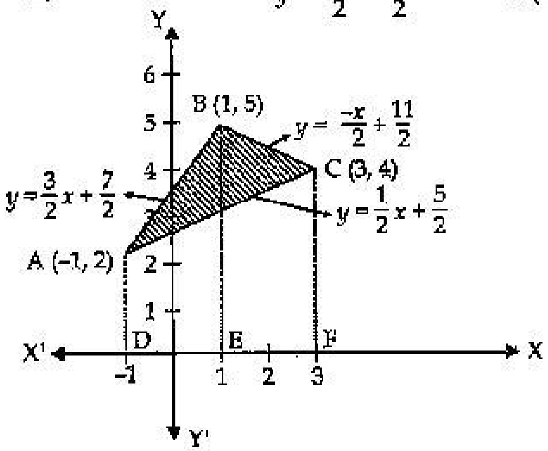
$$y - 4 = \frac{4 - 5}{3 - 1}(x - 3)$$

$$\Rightarrow \quad y = -\frac{1}{2}x + \frac{11}{2} \quad \text{...(ii)}$$

Equation of AC is

$$y - 2 = \frac{4 - 2}{3 + 1}(x + 1)$$

$$\Rightarrow \qquad y = \frac{1}{2}x + \frac{5}{2} \qquad \text{...(iii)}$$



Area of the required triangular region, ABC

= Area of trapezium ADEB + Area of trapezium BEFC – Area of trapezium ADFC

$$= \int_{-1}^{1} y_{AB}\, dx + \int_{1}^{3} y_{BC}\, dx - \int_{-1}^{3} y_{AC}\, dx$$

$$= \int_{-1}^{1}\left(\frac{3}{2}x + \frac{7}{2}\right)dx + \int_{1}^{3}\left(\frac{-1}{2}x + \frac{11}{2}\right)dx - \int_{-1}^{3}\left(\frac{1}{2}x + \frac{5}{2}\right)dx$$

$$= \left[\frac{3}{4}x^2 + \frac{7}{2}x\right]_{-1}^{1} + \left[-\frac{x^2}{4} + \frac{11}{2}x\right]_{1}^{3} - \left[\frac{x^2}{4} + \frac{5}{2}x\right]_{-1}^{3}$$

$$= \left(\frac{3}{4} + \frac{7}{2}\right) - \left(\frac{3}{4} - \frac{7}{2}\right) + \left(-\frac{9}{4} + \frac{33}{2}\right) - \left(\frac{-1}{4} + \frac{11}{2}\right) - \left(\frac{9}{4} + \frac{15}{2}\right) + \left(\frac{1}{4} - \frac{5}{2}\right)$$

$$= \frac{3}{4} + \frac{7}{2} - \frac{3}{4} + \frac{7}{2} - \frac{9}{4} + \frac{33}{2} + \frac{1}{4} - \frac{11}{2} - \frac{9}{4} - \frac{15}{2} + \frac{1}{4} - \frac{5}{2}$$

$= 7 - 4 + 1 = 4$ sq. units **Ans.**

27. Find the equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. Also find the distance of the plane obtained above, from the origin. [6]

Solution : Equation of any plane through the line of intersection of the planes.

$$x + y + z - 1 = 0$$

and
$$2x + 3y + 4z - 5 = 0$$

$$x + y + z - 1 + \lambda(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow (1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z - (1 + 5\lambda) = 0 \qquad \text{...(i)}$$

This plane is perpendicular to the plane

$$x - y + z = 0$$

$$\therefore \quad (1 + 2\lambda) \cdot 1 + (1 + 3\lambda)(-1) + (1 + 4\lambda) \cdot 1 = 0$$

$$\Rightarrow \quad 1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0$$

$$\Rightarrow \quad 1 - 3\lambda = 0$$

$$\Rightarrow \quad \lambda = -\frac{1}{3}$$

Substituting $\lambda = -\frac{1}{3}$ in (i), we get

$$\left(1 - \frac{2}{3}\right)x + 0.y + \left(1 - \frac{4}{3}\right)z - \left(1 - \frac{5}{3}\right) = 0$$

$$\Rightarrow \quad x - z + 2 = 0 \qquad \text{...(ii)}$$

This is the equation of the required plane.

Distance of plane (2) from origin (0, 0, 0)

= Length of ⊥ from (0, 0, 0) on plane

$$= \frac{0 - 0 + 2}{\sqrt{1^2 + (-1)^2}}$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2} \text{ units.} \qquad \textbf{Ans.}$$

OR

Find the distance of the point (2, 12, 5) from the point of intersection of the line $\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0$.

Solution : The given line is

$$\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \qquad \text{...(i)}$$

Writing equation in cartesian form

$$\frac{x - 2}{3} = \frac{y + 4}{4} = \frac{z - 2}{2} = \lambda \text{ (say)}$$

∴ Point on line is $(3\lambda + 2, 4\lambda - 4, 2\lambda + 2)$

This lies on the plane

$$\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0$$

$$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}).(\hat{i} - 2\hat{j} + \hat{k}) = 0$$

$$\Rightarrow \quad x - 2y + z = 0 \qquad \text{...(ii)}$$

$$\therefore \quad 3\lambda + 2 - 2(4\lambda - 4) + 2\lambda + 2 = 0$$

$$\Rightarrow \quad -3\lambda + 12 = 0$$

$$\Rightarrow \quad \lambda = 4$$

∴ The point of intersection of (i) and (ii) is

$$(3 \times 4 + 2, 4 \times 4 - 4, 2 \times 4 + 2) = (14, 12, 10).$$

Distance of the point (2, 12, 5) from the point (14, 12, 10)

$$= \sqrt{(14 - 2)^2 + (12 - 12)^2 + (10 - 5)^2}$$

$$= \sqrt{144 + 0 + 25} = \sqrt{169} = 13 \text{ units} \qquad \textbf{Ans.}$$

28. A manufacturing company makes two types of teaching aids A and B of Mathematics for class

XII. Each type of A requires 9 labour hours of fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ` 80 on each piece of type A and ₹ 120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit ? Make it as an LPP and solve graphically. What is the maximum profit per week ? **[6]**

Solution : Let x and y be the number of teaching aids of type A and B respectively. Then the LPP is

Maximize $Z = 80x + 120y$

Subject to constraints :

$$9x + 12y \leq 180$$

$$x + 3y \leq 30$$

and $x \geq 0,\ y \geq 0,$

First we draw the lines AB and CD whose equations are

$$9x + 12y = 180$$

$$\Rightarrow \quad 3x + 4y = 60 \qquad ...(i)$$

x	20	0
y	0	15

and $x + 3y = 30$ ...(ii)

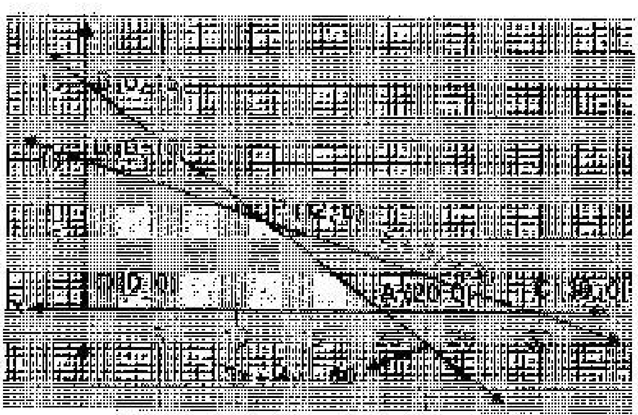
x	30	0
y	0	10

Let P be the point of intersection of the lines

$$3x + 4y = 60$$

and $x + 3y = 30$

Solving these equations, we get point P(12, 6).



The feasible region is OAPDO which is shaded in the figure.

The vertices of the feasible region are O (0, 0), A (20, 0), P(12, 6) and D (0, 10)

The value of objective function

$Z = 80x + 120y$ as follows :

Corner Points	Maximize $Z = 80x + 120y$
At O (0, 0)	0
At A (20, 0)	1600
At P (12, 6)	960 + 720 = 1680 maximum
At D (0, 10)	1200

∴ The profit is maximum at P(12, 6) *i.e.*, when the teaching aids of types A and B are 12 and 6 respectively.

Also maximum profit = ₹ 1680 per week. **Ans.**

29. There are three coins. One is a two-headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin ? **[6]**

Solution : Let A be the two headed coin, B be the biased coin showing up heads 75% of the times and C be the biased coin showing up tails 40% (*i.e.*, showing up heads 60%) of the times.

Let E_1, E_2 and E_3 be the events of choosing coins of the type A, B, C respectively. Let S be the event of getting a head. Then,

$$P(E_1) = \frac{1}{3}, P(E_2) = \frac{1}{3}, P(E_3) = \frac{1}{3}$$

$$P(S/E_1) = 1, P(S/E_2) = 75\% = \frac{75}{100} = \frac{3}{4},$$

$$P(S/E_3) = 60\% = \frac{60}{100} = \frac{3}{5}.$$

∴ By Bayes' theorem, the required probability

$$P(E_1/S) = \frac{P(E_1).P(S/E_1)}{\sum_{i=1}^{3} P(E_i).P(S/E_i)}$$

$$= \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times 1+\frac{1}{3}\times\frac{3}{4}+\frac{1}{3}\times\frac{3}{5}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3}\left(1+\frac{3}{4}+\frac{3}{5}\right)}$$

$$= \frac{1}{1+\frac{3}{4}+\frac{3}{5}}$$

$$= \frac{20}{20+15+12} = \frac{20}{47}. \quad \text{Ans.}$$

OR

Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find the probability distribution of the random variable X, and hence find the mean of the distribution.

Solution : Sample space

$$S = \{1, 2, 3, 4, 5, 6\}$$

Number of ways of selecting any two members of S is

$${}^6C_2 = \frac{6\times5}{2!} = 15.$$

Now X denotes the larger of the two selected numbers.

$\therefore$ $P(X=1) = 0$

$P(X=2) = \frac{1}{15}$ $(2>1)$

$P(X=3) = \frac{2}{15}$ $(3>1, 3>2)$

$P(X=4) = \frac{3}{15}$ $(4>1, 4>2, 4>3)$

$P(X=5) = \frac{4}{15}$ $(5>1, 5>2, 5>3, 5>4)$

$P(X=6) = \frac{5}{15}$ $(6>1, 6>2, 6>3, 6>4, 6>5)$

$\therefore$ The probability distribution is

X	1	2	3	4	5	6
$P(X=x)$	0	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

$\therefore$ Mean of the distribution $= \sum_{i=1}^{6} P_i X_i$

$$= 0\times1+\frac{1}{15}\times2+\frac{2}{15}\times3+\frac{3}{15}\times4+\frac{4}{15}\times5+\frac{5}{15}\times6$$

$$= \frac{1}{15}(0+2+6+12+20+30)$$

$$= \frac{70}{15} = \frac{14}{3}. \quad \text{Ans.}$$

●●

Mathematics 2014 (Outside Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — A

9. Evaluate : $\int_e^{e^2} \frac{dx}{x\log x}$ **[1]**

Solution : Let, $I = \int_e^{e^2} \frac{dx}{x\log x}$

Putting, $\log x = t$

$\Rightarrow$ $\frac{1}{x}dx = dt$

Also, when $x = e$

$\Rightarrow$ $t = \log e = 1$

and $x = e^2$

$\Rightarrow$ $t = \log e^2 = 2\log e = 2$

$\therefore$ $I = \int_1^2 \frac{dt}{t}$

$= [\log t]_1^2$

$= \log 2 - \log 1 = \log 2$ **Ans.**

10. Find a vector $\vec{a}$ of magnitude $5\sqrt{2}$, making an angle of $\frac{\pi}{4}$ with x-axis, $\frac{\pi}{2}$ with y-axis and an acute angle θ with z-axis. **[1]**

Solution : Here,

$$l = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

$$m = \cos\frac{\pi}{2} = 0, n = \cos\theta$$

Since, $l^2 + m^2 + n^2 = 1$

$\Rightarrow$ $\frac{1}{2}+0+\cos^2\theta = 1$

$\Rightarrow$ $\cos^2\theta = \frac{1}{2}$

$\Rightarrow$ $\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\Rightarrow$ $\cos\theta = \frac{1}{\sqrt{2}}$ (Rejected –ve as θ is acute)

$$\Rightarrow \qquad \theta = \frac{\pi}{4}$$

$\therefore$ The vector of magnitude $5\sqrt{2}$ is

$$\vec{a} = 5\sqrt{2}(l\hat{i}+m\hat{j}+n\hat{k})$$

$$= 5\sqrt{2}\left(\frac{1}{\sqrt{2}}\hat{i}+0\hat{j}+\frac{1}{\sqrt{2}}\hat{k}\right)$$

$$= 5(\hat{i}+\hat{k})$$ **Ans.**

SECTION — B

19. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$ **[4]**

Solution : Taking L. H. S.

Let,
$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$ and take 2 common from C_1, we get

$$\Delta = 2\begin{vmatrix} a+b+c & c+a & a+b \\ p+q+r & r+p & p+q \\ x+y+z & z+x & x+y \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$, we get

$$\Delta = 2\begin{vmatrix} a+b+c & -b & -c \\ p+q+r & -q & -r \\ x+y+z & -y & -z \end{vmatrix}$$

Taking -1 common from C_2 and C_3

$$= 2(-1)(-1)\begin{vmatrix} a+b+c & b & c \\ p+q+r & q & r \\ x+y+z & y & z \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2 - C_3$, we get

$$\Delta = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = \text{R.H.S.}$$

Hence Proved.

20. If $x = a \sin 2t\ (1 + \cos 2t)$ and $y = b \cos 2t\ (1 - \cos 2t)$, show that at $t = \frac{\pi}{4}, \left(\frac{dy}{dx}\right) = \frac{b}{a}$. **[4]**

Solution : Here,

$$x = a \sin 2t\ (1 + \cos 2t) \quad \text{(i)}$$

$$y = b \cos 2t\ (1 - \cos 2t) \quad \text{(ii)}$$

Differentiating (i) w. r. t. 't', we get

$$\Rightarrow \frac{dx}{dt} = a[2\cos 2t(1+\cos 2t) + \sin 2t.(-2\sin 2t)]$$

Differentiating (ii) w. r. t. 't', we get

$$\frac{dy}{dt} = b[-2\sin 2t(1-\cos 2t) + \cos 2t.2\sin 2t]$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{2b[-\sin 2t + 2\sin 2t \cos 2t]}{2a[\cos 2t + \cos^2 2t - \sin^2 2t]}$$

Putting $t = \frac{\pi}{4}$, we get

$$\frac{dy}{dx} = \frac{b}{a}\left[\frac{-\sin\frac{\pi}{2} + 2\sin\frac{\pi}{2}\cos\frac{\pi}{2}}{\cos\frac{\pi}{2} + \cos^2\frac{\pi}{2} - \sin^2\frac{\pi}{2}}\right]$$

$$= \frac{b}{a}\frac{[-1+0]}{[0+0-1]}$$

$$\therefore \frac{dy}{dx} = \frac{b}{a}$$ **Hence Proved.**

21. Find the particular solution of the differential equation $x(1 + y^2)\, dx - y(1 + x^2)\, dy = 0$, given that $y = 1$ when $x = 0$. **[4]**

Solution : The given differential equation is

$$x(1+y^2)dx - y(1+x^2)\,dy = 0 \qquad \text{...(i)}$$

Separate the given differential equation, we get

$$\frac{x}{1+x^2}dx - \frac{y}{1+y^2}dy = 0$$

$$\Rightarrow \frac{2x}{1+x^2}dx - \frac{2y}{1+y^2}dy = 0$$

On integrating, we get

$$\int\frac{2y}{1+y^2}dy - \int\frac{2x}{1+x^2}dx = \text{constant}$$

$$\Rightarrow \log(1+y^2) - \log(1+x^2) = \log C$$

$$\Rightarrow \log\frac{1+y^2}{1+x^2} = \log C$$

$$\Rightarrow 1+y^2 = C(1+x^2)$$

Putting $y = 1$ and $x = 0$, we get

$$1 + 1 = C(1+0)$$

$$\Rightarrow C = 2$$

$\therefore$ The required particular solution of equation (i) is

$$1+y^2 = 2(1+x^2)$$ **Ans.**

22. Find the vector and Cartesian equations of the line passing through the point (2, 1, 3) and

perpendicular to the lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$. **[4]**

Solution : Let the equation of any line passing through (2, 1, 3) and perpendicular to the lines

$$\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$$

and $$\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$$

be $$\frac{x-2}{l}=\frac{y-1}{m}=\frac{z-3}{n} \quad \text{...(i)}$$

$$\therefore \quad l.1+m.2+n.3=0 \quad \text{...(ii)}$$

$$l(-3)+m.2+n.5=0 \quad \text{...(iii)}$$

Solving equations (ii) and (iii), we get

$$\frac{l}{10-6}=\frac{m}{-9-5}=\frac{n}{2+6}$$

$$\Rightarrow \quad \frac{l}{2}=\frac{m}{-7}=\frac{n}{4}$$

$\therefore$ The equation of the required line is

$$\frac{x-2}{2}=\frac{y-1}{-7}=\frac{z-3}{4}$$

Also its vector equation is

$$\vec{r}=(2\hat{i}+\hat{j}+3\hat{k})+\lambda(2\hat{i}-7\hat{j}+4\hat{k})$$

Ans.

SECTION — C

28. Evaluate : $\int\left(\sqrt{\cot x}+\sqrt{\tan x}\right)dx$. **[6]**

Solution : Let $\quad I=\int\left(\sqrt{\cot x}+\sqrt{\tan x}\right)dx$

Putting $\quad \sqrt{\tan x}=t$

$$\Rightarrow \quad \tan x=t^2$$

$$\Rightarrow \quad \sec^2 x\,dx=2t\,dt$$

$$\Rightarrow \quad dx=\frac{2t\,dt}{1+\tan^2 x}$$

$(\because \sec^2 x=1+\tan^2 x)$

$$=\frac{2t}{1+t^4}dt$$

$$\therefore \quad I=\int\left(\frac{1}{t}+t\right).\frac{2t}{1+t^4}dt$$

$$=2\int\frac{t^2+1}{t^4+1}dt$$

$$2\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt=2\int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt$$

Again putting, $t-\frac{1}{t}=y$

$$\Rightarrow \quad \left(1+\frac{1}{t^2}\right)dt=dy$$

$$\Rightarrow \quad I=2\int\frac{dy}{y^2+\left(\sqrt{2}\right)^2}+C$$

$$=2.\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right)+C$$

$$=\sqrt{2}\tan^{-1}\left(t-\frac{1}{t}\right)\Big/\sqrt{2}+C$$

$$=\sqrt{2}\tan^{-1}\left[\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right]+C. \quad \textbf{Ans.}$$

29. Prove that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. **[6]**

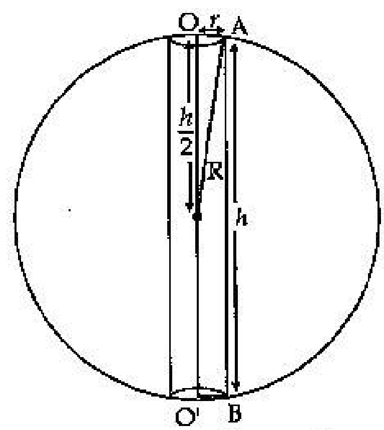
Solution : From the figure,

$$\left(\frac{h}{2}\right)^2+r^2=R^2$$

$$\Rightarrow \quad r^2=R^2-\frac{h^2}{4} \quad \text{...(i)}$$

Now, V = Volume of the cylinder inscribed in a sphere

$$\therefore \quad V=\pi r^2 h$$



$$=\pi h\left(R^2-\frac{h^2}{4}\right) \quad \text{[using (i)]}$$

$$\therefore \quad V=\pi\left(R^2 h-\frac{h^3}{4}\right)$$

Differentiating w. r. t. h, we get

$$\frac{dV}{dh}=\pi\left(R^2-\frac{3h^2}{4}\right) \quad \text{...(ii)}$$

and $$\frac{d^2V}{dh^2}=\pi\left(0-\frac{3}{4}.2h\right) \quad \text{...(iii)}$$

For maxima or minima

$$\frac{dV}{dh} = 0$$

From (ii),

$$R^2 - \frac{3}{4}h^2 = 0$$

$$\Rightarrow \quad h^2 = \frac{4}{3}R^2$$

$$\Rightarrow \quad h = \frac{2R}{\sqrt{3}}$$

For the value of h, from (iii),

$$\frac{d^2V}{dh^2} = -\frac{3}{2}\pi\frac{2R}{\sqrt{3}} = -\sqrt{3}.\pi R < 0(-\text{ve})$$

$\rightarrow$ V is maximum.

Also maximum value of V

$$= \pi.\frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4}.\frac{4}{3}R^2\right)$$

$$= \pi.\frac{2R}{\sqrt{3}}.\frac{2}{3}R^2 = \frac{4\pi}{3\sqrt{3}}R^3$$

$$= \frac{4\sqrt{3}}{9}\pi R^3 \text{cu.units}$$ **Ans.**

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Mathematics 2014 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

> **Note : Except for the following questions, all the remaining questions have been asked in previous sets.**

SECTION — A

9. If $\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$, find the value of a. **[1]**

Solution : Given,

$$\int_0^a \frac{1}{4+x^2}dx = \frac{\pi}{8}$$

$$\Rightarrow \quad \int_0^a \frac{1}{x^2+2^2}dx = \frac{\pi}{8}$$

$$\Rightarrow \quad \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^a = \frac{\pi}{8}$$

$$\left(\because \int\frac{1}{x^2+a^2} = \frac{1}{2}\tan^{-1}\frac{x}{a}\right)$$

$$\Rightarrow \quad \frac{1}{2}\tan^{-1}\frac{a}{2} = \frac{\pi}{8}$$

$$\Rightarrow \quad \tan^{-1}\frac{a}{2} = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{a}{2} = \tan\frac{\pi}{4}$$

$$\Rightarrow \quad \frac{a}{2} = 1,$$

$$\Rightarrow \quad a = 2$$ **Ans.**

10. If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, $|\vec{a}+\vec{b}| = 13$ and $|\vec{a}| = 5$, find the value of $|\vec{b}|$. **[1]**

Solution : Given, $|\vec{a}| = 5, |\vec{a}+\vec{b}| = 13$

Since $\vec{a}$ and $\vec{b}$ are perpendicular.

$$\therefore \quad \vec{a}.\vec{b} = 0$$

Now $$|\vec{a}+\vec{b}| = 13$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = 13^2$$

$$\Rightarrow \quad (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 169$$

$$\Rightarrow \quad \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b} = 169$$

$$\Rightarrow \quad |\vec{a}|^2 + 0 + 0 + |\vec{b}|^2 = 169$$

$$(\because \vec{a}.\vec{b} = \vec{b}.\vec{a} = 0)$$

$$|\vec{b}|^2 = 169 - |\vec{a}|^2$$

$$= 169 - 5^2$$

$$= 169 - 25 = 144$$

$$\Rightarrow \quad |\vec{b}| = 12.$$ **Ans.**

SECTION–B

19. Using properties of determinants, prove that :

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc+bc+ca+ab$$ **[4]**

Solution : Taking L. H. S.

$$\text{Let } \Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

Applying $R_1 \to \frac{R_1}{a}, R_2 \to \frac{R_2}{b}$ and $\to \frac{R_3}{c}$, we get

$$\Delta = abc\begin{vmatrix} \frac{1+a}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1+b}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1+c}{c} \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$ and take $1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ common from R_1, we get

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we get

$$\Delta = (abc+bc+ca+ab)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

Expanding along R_1, we get

$$= (abc + bc + ca + ab)1.(1-0)$$

$$= abc + bc + ca + ab = \text{R. H. S.}$$

Hence Proved.

20. If $x = \cos t\,(3 - 2\cos^2 t)$ and $y = \sin t\,(3 - 2\sin^2 t)$, find the value of $\frac{dy}{dx}$ at $t = \frac{\pi}{4}$. [4]

Solution : Here, $x = \cos t\,(3 - 2\cos^2 t)$ ...(i)

$y = \sin t\,(3 - 2\sin^2 t)$ ...(ii)

Differentiating (i) w.r.t. t, we get

$$\frac{dx}{dt} = -\sin t\,(3 - 2\cos^2 t) + \cos t\,[2.2\cos t \sin t]$$

$$= -3\sin t + 6\cos^2 t \sin t$$

Differentiating (ii) w.r.t. t, we get

$$\frac{dy}{dt} = \cos t\,(3 - 2\sin^2 t) + \sin t\,(-2.2\sin t\cos t)$$

$$= 3\cos t - 6\sin^2 t\cos t$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{3\cos t - 6\sin^2 t\cos t}{-3\sin t + 6\cos^2 t\sin t}$$

$$= \frac{3\cos t(1-2\sin^2 t)}{3\sin t(2\cos^2 t - 1)}$$

$$= \frac{\cos t.\cos 2t}{\sin t.\cos 2t} = \cot t$$

$$(\because 1 - 2\sin^2 t = 2\cos^2 t - 1 = \cos 2t)$$

Put $t = \frac{\pi}{4}$

$\therefore \frac{dy}{dx} = \cot\frac{\pi}{4} = 1$ **Ans.**

21. Find the particular solution of the differential equation $\log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0$ when $x = 0$. [4]

Solution : The given differential equation is

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y$$

$$\frac{dy}{dx} = e^{3x+4y} = e^{3x}.e^{4y}$$

$$\Rightarrow \quad e^{-4y}dy = e^{3x}dx$$

On integrating, we get

$$\int e^{3x}dx - \int e^{-4y}dy = C$$

$$\Rightarrow \quad \frac{e^{3x}}{3} - \frac{e^{-4y}}{-4} = C$$

$$\Rightarrow \quad \frac{e^{3x}}{3} + \frac{e^{-4y}}{4} = C$$

Putting, $y = 0$ when $x = 0$

$$\therefore \quad \frac{1}{3} + \frac{1}{4} = C$$

$$\Rightarrow \quad C = \frac{7}{12}$$

Hence, $\frac{e^{3x}}{3} + \frac{e^{-4y}}{4} = \frac{7}{12}$

$\therefore$ The required particular solution of the given differential equation is $4e^{3x} + 3e^{-4y} = 7$ **Ans.**

22. Find the value of p, so that the lines $l_1 : \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$ and $l_2 : \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$ are perpendicular to each other. Also find the equations of a line passing through a point (3, 2, – 4) and parallel to line l_1. [4]

Solution : The given lines are

$$l_1 : \frac{1-x}{3} = \frac{7y-14}{p} = \frac{z-3}{2}$$

$$\Rightarrow \quad l_1 : \frac{x-1}{-3} = \frac{y-2}{p/7} = \frac{z-3}{2} \quad ...(i)$$

and $$l_2 : \frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

$$\Rightarrow \quad l_2 : \frac{x-1}{-3p/7} = \frac{y-5}{1} = \frac{z-6}{-5} \quad ...(ii)$$

Since l_1 and l_2 are perpendicular

$$\therefore \quad (-3).\left(-\frac{3p}{7}\right)+\left(\frac{p}{7}\right).1+2.(-5) = 0$$

$$\Rightarrow \quad \frac{10p}{7}-10 = 0$$

$$\Rightarrow \quad p = 7$$

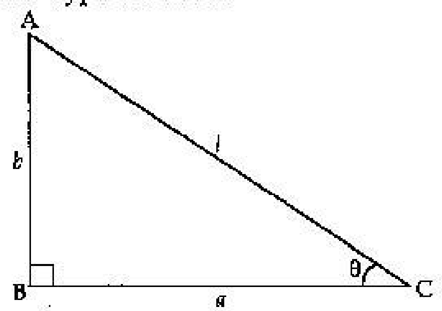
$\therefore$ Equation of the line passes through (3, 2, – 4) and parallel to l_1 is

$$\frac{x-3}{-3} = \frac{y-2}{1} = \frac{z+4}{2} \qquad \textbf{Ans.}$$

SECTION–C

28. If the sum of the lengths of the hypotenuse and a side of a right triangle is given, show that the area of the triangle is maximum, when the angle of between them is 60°. [6]

Solution : Let Δ ABC be right angled with side a and hypotenuse l be



$$a + l = p \qquad \text{(given)}$$

$$\Rightarrow \quad l = p - a$$

Let θ be the angle between them.

Now, $\quad A = \text{Area of } \Delta \text{ ABC}$

$$= \frac{1}{2}ab = \frac{1}{2}a\sqrt{l^2 - a^2}$$

$$(\because b^2 = l^2 - a^2)$$

$$= \frac{1}{2}a\sqrt{(p-a)^2 - a^2}$$

$$\therefore \quad A = \frac{1}{2}a\sqrt{p^2 - 2pa}$$

Let $\quad z = A^2 = \frac{1}{4}a^2(p^2 - 2pa)$

Diff. w. r. t. a, $\quad \frac{dz}{da} = \frac{1}{4}(2ap^2 - 6pa^2) \quad ...(i)$

$$= \frac{1}{4}.2ap(p-3a)$$

For max. or min

$$\frac{dz}{da} = 0$$

$$\Rightarrow \quad p - 3a = 0$$

$$\Rightarrow \quad a = \frac{p}{3}$$

Again differentiate eq. (i) w. r. t. a,

$$\frac{d^2z}{da^2}\ a = \frac{p}{3} = \frac{1}{4}(2p^2 - 12pa)$$

$$\frac{d^2z}{da^2} = \frac{1}{4}\left(2p^2 - 12p.\frac{p}{3}\right)$$

$$= \frac{1}{4}(2p^2 - 4p^2) = \frac{-2p^2}{4} = \frac{-p^2}{2} < 0$$

$\Rightarrow$ Area, A is maximum.

Now from the figure,

$$\cos\theta = \frac{a}{l} = \frac{a}{p-a} = \frac{a}{3a-a} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = 60°$$

Thus area A is maximum, when angle between the hypotenuse and a side is 60°. **Hence Proved.**

29. Evaluate :

$$\int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x}dx. \qquad [6]$$

Solution : Let

$$I = \int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x}dx.$$

Dividing by $\cos^4 x$ in Nr and Dr, we get

$$= \int \frac{\sec^4 x\, dx}{\tan^4 x + \tan^2 x + 1}$$

$$= \int \frac{\sec^2 x . \sec^2 x}{\tan^4 x + \tan^2 x + 1}dx$$

$$= \int \frac{(1+\tan^2 x)\sec^2 x\, dx}{\tan^4 x + \tan^2 x + 1}$$

$$(\because \sec^2 x = 1 + \tan^2 x)$$

Putting $\tan x = t$

$$\Rightarrow \quad \sec^2 x\, dx = dt$$

$$\therefore \quad I = \int \frac{1+t^2}{t^4 + t^2 + 1}dt$$

$$= \int \frac{1+\frac{1}{t^2}}{t^2+1+\frac{1}{t^2}}dt = \int \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+3}dt$$

Again putting $t - \frac{1}{t} = y$

$$\Rightarrow \quad \left(1+\frac{1}{t^2}\right)dt = dy$$

Thus, $\quad I = \int \frac{dy}{y^2 + (\sqrt{3})^2}$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{y}{\sqrt{3}}\right) + C$$

$$\left(\because \int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+C\right)$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(t-\frac{1}{t}\right)/\sqrt{3}+C$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{\tan x - \cot x}{\sqrt{3}}\right)+C \quad \textbf{Ans.}$$

Mathematics 2014 (Delhi) — SET I

Time allowed : 3 hours — **Maximum marks : 100**

SECTION — A

1. Let * be binary operation, on the set of all non-zero real numbers, given by $a * b = \frac{ab}{5}$ for all $a, b \in R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.** **[1]**

2. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x. **[1]**

Solution : Given,

$$\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right) = 1$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5}+\cos^{-1}x = \sin^{-1} \quad (1)$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5}+\cos^{-1}x = \frac{\pi}{2}$$

$$\left(\because \sin^{-1}(1) = \frac{\pi}{2}\right)$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \frac{\pi}{2}-\cos^{-1}x$$

$$\Rightarrow \quad \sin^{-1}\frac{1}{5} = \sin^{-1}x$$

$$\left(\because \cos^{-1}x+\sin^{-1}x = \frac{\pi}{2}\right)$$

$$\therefore \quad x = \frac{1}{5} \quad \textbf{Ans.}$$

3. If $2\begin{bmatrix}3 & 4\\5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$, find $(x-y)$. **[1]**

Solution : Given,

$$2\begin{bmatrix}3 & 4\\5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix} = \begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}6 & 8\\10 & 2x\end{bmatrix}+\begin{bmatrix}1 & y\\0 & 1\end{bmatrix} = \begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}6+1 & 8+y\\10+0 & 2x+1\end{bmatrix} = \begin{bmatrix}7 & 0\\10 & 5\end{bmatrix}$$

Equating the corresponding elements, we get

$$\therefore \quad 8+y = 0$$

$$\Rightarrow \quad y = -8$$

$$\text{and} \quad 2x+1 = 5$$

$$\Rightarrow \quad 2x = 4$$

$$\Rightarrow \quad x = 2$$

$$\therefore \quad x-y = 2-(-8)$$

$$= 2+8 = 10 \quad \textbf{Ans.}$$

4. Solve the following matrix equation for $x : [x \;\; 1]\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix} = 0$. **[1]**

Solution : Given,

$$[x \;\; 1]\begin{bmatrix}1 & 0\\-2 & 0\end{bmatrix} = 0$$

$$\Rightarrow \quad [x-2 \;\; 0+0] = [0 \; 0]$$

$$\Rightarrow \quad x-2 = 0$$

$$\therefore \quad x = 2 \quad \textbf{Ans.}$$

5. If $\begin{vmatrix}2x & 5\\8 & x\end{vmatrix} = \begin{vmatrix}6 & -2\\7 & 3\end{vmatrix}$, write the value of x. **[1]**

Solution : Given,

$$\begin{vmatrix}2x & 5\\8 & x\end{vmatrix} = \begin{vmatrix}6 & -2\\7 & 3\end{vmatrix}$$

$$\Rightarrow \quad 2x^2-40 = 18-(-14)$$

$$\Rightarrow \quad 2x^2 = 18+14+40$$

$$\Rightarrow \quad 2x^2 = 72$$

$$\Rightarrow \quad x^2 = 36$$

$$\therefore \quad x = \pm 6 \quad \textbf{Ans.}$$

6. Write the antiderivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$. **[1]**

Solution : The antiderivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$

**Answer is not given due to the change in present syllabus

$$= \int\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)dx$$

(Antiderivative = Integral)

$$= 3\int x^{1/2}dx+\int x^{-1/2}dx$$

$$= 3.\frac{x^{3/2}}{3/2}+\frac{x^{1/2}}{1/2}+C$$

$$= 2x^{3/2}+2x^{1/2}+C$$

$$= 2x\sqrt{x}+2\sqrt{x}+C$$

$$= 2\sqrt{x}(x+1)+C. \quad \textbf{Ans.}$$

7. Evaluate : $\int_0^3 \frac{dx}{9+x^2}$ **[1]**

Solution : $\int_0^3 \frac{dx}{9+x^2} = \int_0^3 \frac{dx}{x^2+3^2}$

$$= \left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$$

$$\left(\because \int\frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}\right)$$

$$= \frac{1}{3}\left[\tan^{-1}1-\tan^{-1}0\right]$$

$$= \frac{1}{3}\left[\frac{\pi}{4}-0\right]=\frac{\pi}{12}. \quad \textbf{Ans.}$$

8. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}-3\hat{j}+6\hat{k}$. **[1]**

Solution: Let $\vec{a}=\hat{i}+3\hat{j}+7\hat{k}$ and $\vec{b}=2\hat{i}-3\hat{j}+6\hat{k}$

$\therefore$ Projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$

$$= \frac{(\hat{i}+3\hat{j}+7\hat{k}).(2\hat{i}-3\hat{j}+6\hat{k})}{\sqrt{2^2+(-3)^2+6^2}}$$

$$= \frac{1\times2+3\times(-3)+7\times6}{\sqrt{4+9+36}}$$

$$= \frac{2-9+42}{7}=\frac{35}{7}=5. \quad \textbf{Ans.}$$

9. If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a}+\vec{b}$ is also a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$. [1]

Solution : Given, $|\vec{a}|=1, |\vec{b}|=1$

and $|\vec{a}+\vec{b}| = 1$

$\Rightarrow$ $|\vec{a}+\vec{b}|^2 = 1$

$\Rightarrow$ $(\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$

$\Rightarrow$ $\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b} = 1$

$\Rightarrow$ $|\vec{a}|^2+2\vec{a}.\vec{b}+|\vec{b}|^2 = 1$

$\Rightarrow$ $1+2\vec{a}.\vec{b}+1 = 1$

$\Rightarrow$ $2\vec{a}.\vec{b} = 1$

$\Rightarrow$ $2|\vec{a}|.|\vec{b}|\cos\theta = -1$

$\Rightarrow$ $2.1.1\cos\theta = -1$

$\Rightarrow$ $\cos\theta = -\frac{1}{2}$

$\Rightarrow$ $\cos\theta = \cos 120^\circ$

$\therefore$ $\theta = 120^\circ$ **Ans.**

10. Write the vector equation of the plane, passing through the point (a, b, c) and parallel to the plane $\vec{r}.(\hat{i}+\hat{j}+\hat{k})=2$. **[1]**

Solution : The given plane is

$$\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = 2 \quad ...(i)$$

$\Rightarrow$ $(x\hat{i}+y\hat{j}+z\hat{k}).(\hat{i}+\hat{j}+\hat{k}) = 2$

$\Rightarrow$ $x+y+z = 2$ ...(ii)

$\therefore$ Equation of a plane parallel to (ii), is

$$x+y+z = \lambda \quad ...(iii)$$

The plane is passing through (a, b, c)

$\therefore$ $\lambda = a+b+c$

$\therefore$ The vector equation of the required plane is

$$\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = a+b+c. \textbf{Ans.}$$

SECTION — B

11. Let A = {1, 2 ,3, ..., 9} and R be relation in A × A defined by (a, b) R (c, d) if $a+d=b+c$ for $(a,b),(c,d)$ in A × A. Prove that R is an equivalence relation. Also obtain the equivalence class [(2, 5)]. **[4]**

Solution : Here, A = {1, 2, 3, ..., 9} and R is a relation on A × A defined by

(a, b) R $(c, d) \Leftrightarrow a+d=b+c \, \forall a, b, c, d \in A$

(i $\forall(a,b) \in A\times A$

$a+b = b+a$

$\Rightarrow (a,b)R(a,b) \ \forall(a,b)\in A\times A$

$\Rightarrow$ R is reflexive on A.

(ii) Let (a, b) R (c, d)

$\Rightarrow$ $a+d=b+c$

$\Rightarrow$ $b+c=a+d$

$\Rightarrow$ $c+b=d+a$

$\Rightarrow \quad (c, d)\ R\ (a, b)$

$\Rightarrow$ R is symmetric on A.

(iii) Let $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow \quad a + d = b + c$ and $c + f = d + e$

$\Rightarrow (a + d) + (c + f) = (b + c) + (d + e)$

$\Rightarrow \quad a + f = b + e$

$\Rightarrow \quad (a, b)\ R\ (e, f)$

$\Rightarrow$ R is transitive on A.

Hence R is an equivalence relation on A.

Also equivalence class $[(2, 5)]$

$$= \{(a, b) \in A \times A \mid (2,5)\ R\ (a, b)\}$$

$$= \{(a, b) \in A \times A \mid 2 + b = 5 + a\}$$

$$= \{(a, b) \in A \times A \mid b = a + 3\}$$

$$= \{(a, a + 3) \mid a \in A\}. \quad \textbf{Ans.}$$

12. Prove that

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2};\ x \in \left(0, \frac{\pi}{4}\right)$$

[4]

Solution : L. H. S.

$$= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right);\ x \in \left(0, \frac{\pi}{4}\right)$$

$$= \cot^{-1}\left(\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})}\right.$$

$$\left.\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})}\right)$$

$$= \cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{(1+\sin x)(1-\sin x)}}{(1+\sin x)-(1-\sin x)}\right)$$

$$= \cot^{-1}\left(\frac{2+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2(1+\cos x)}{2\sin x}\right)$$

$$= \cot^{-1}\left(\frac{4\cos^2 x/2}{4\sin x/2 . \cos x/2}\right)$$

$$\left(\because 1+\cos x = 2\cos^2\frac{x}{2} \text{ and } \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$= \cot^{-1}\left(\frac{\cos x/2}{\sin x/2}\right)$$

$= \cot^{-1}(\cot x/2) = x/2 =$ R.H.S. **Hence Proved.**

OR

Prove that :

$$2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}.$$

Solution : L. H. S.

$$= 2\tan^{-1}\left(\frac{1}{5}\right)+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)+2\tan^{-1}\left(\frac{1}{8}\right)$$

$$= 2\left[\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)\right]+\sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$$

$$= 2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right)+\tan^{-1}\left(\sqrt{\left(\frac{5\sqrt{2}}{7}\right)^2-1}\right)$$

$$\left[\begin{array}{l}\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\\ \because \sec^{-1}x=\tan^{-1}\sqrt{x^2-1}\end{array}\right]$$

$$= 2\tan^{-1}\left(\frac{13/40}{39/40}\right)+\tan^{-1}\left(\sqrt{\frac{1}{49}}\right)$$

$$= 2\tan^{-1}\left(\frac{13}{39}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= 2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left[\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right]+\tan^{-1}\left(\frac{1}{7}\right)$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4} = \text{R. H. S.}$$ **Hence Proved.**

13. Using properties of determinants, prove that

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3$$ **[4]**

Solution : Taking L. H. S.

Let $$\Delta = \begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

Taking $(x+y+z)$ common from R_1,

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_2$, we get

$$\Delta = (x+y+z)\begin{vmatrix} 1 & 0 & 0 \\ 2z & 0 & -(x+y+z) \\ x-y-z & x+y+z & 0 \end{vmatrix}$$

By expanding along R_1, we get

$$(x+y+z).1.(x+y+z)^2$$

$$= (x+y+z)^3 = \text{R. H. S.}$$ **Hence Proved.**

14. Differentiate $\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ with respect to $\cos^{-1}(2x\sqrt{1-x^2})$, when $x \neq 0$. **[4]**

Solution : Let $y = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$

Putting $x = \cos\theta$, we get

$$y = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$(\because 1-\cos^2\theta = \sin^2\theta)$

$$= \tan^{-1}(\tan\theta)$$

$$\Rightarrow \quad y = \theta$$

Differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = 1 \quad \text{...(i)}$$

and let $t = \cos^{-1}(2x\sqrt{1-x^2})$ $(x \neq 0)$

Put $x = \cos\theta$, we get

$$t = \cos^{-1}(2\cos\theta\sqrt{1-\cos^2\theta})$$

$$= \cos^{-1}(2\cos\theta\sin\theta)$$

$$= \cos^{-1}(\sin 2\theta)$$

$$= \cos^{-1}\left[\cos\left(\frac{\pi}{2}-2\theta\right)\right]$$

$$\therefore \quad t = \frac{\pi}{2} - 2\theta$$

Differentiating w. r. t. θ, we get

$$\frac{dt}{d\theta} = 0-2 = -2 \quad \text{...(ii)}$$

Now, $$\frac{dy}{dt} = \frac{dy/d\theta}{dt/d\theta} \quad \text{[Using (i) and (ii)]}$$

$$= -\frac{1}{2}$$ **Ans.**

15. If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$. **[4]**

Solution : Given, $y = x^x$

Taking log on both sides, we get

$$\log y = x \log x$$

Differentiating both sides w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = x.\frac{1}{x} + 1.\log x = 1 + \log x$$

$$\Rightarrow \quad \frac{dy}{dx} = y(1+\log x)$$

Again differentiating, w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{dy}{dx}.(1+\log x) + y.\frac{1}{x}$$

$$= \frac{dy}{dx}\left[\frac{dy}{dx}.\frac{1}{y}\right] + \frac{y}{x}$$

[Using (i)]

$$\Rightarrow \quad \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0.$$ **Hence Proved.**

16. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is **[4]**

(a) strictly increasing

(b) strictly decreasing

Solution : Here

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$$\Rightarrow \quad f'(x) = 12x^3 - 12x^2 - 24x$$

$= 12x(x^2 - x - 2)$
$= 12x[x^2 - 2x + x - 2]$
$= 12x[x(x-2) + 1(x-2)]$
$= 12x(x+1)(x-2)$

The critical lvalues for f are -1, 0 and 2

$$\underset{-\infty}{\longleftarrow} \overset{-}{\quad} \underset{-1}{|} \overset{+}{\quad} \underset{0}{|} \overset{-}{\quad} \underset{2}{|} \overset{+}{\quad} \underset{\infty}{\longrightarrow}$$

Intervals	Test value	sign of $f'(x) = 12x(x+1)(x-2)$	Nature of function $f(x)$
$(-\infty, -1)$	$x = -1.5$	$(-)(-)(-) = - < 0$	Strictly decreasing
$(-1, 0)$	$x = -0.5$	$(-)(-)(-) = + > 0$	Strictly increasing
$(0, 2)$	$x = 1$	$(+)(+)(-) = - < 0$	Strictly decreasing
$(2, \infty)$	$x = 3$	$(+)(+)(+) = + > 0$	Strictly increasing

$\therefore f$ is strictly increasing in $(-1, 0) \cup (2, \infty)$ and strictly decreasing in $(-\infty, -1) \cup (0, 2)$. **Ans.**

OR

Find the equations of the tangent and normal to the curve $x = a\sin^3\theta$ and $y = a\cos^3\theta$ at $\theta = \frac{\pi}{4}$.

Solution : The given curve is $x = a\sin^3\theta$; $y = a\cos^3\theta$ ...(i)

At, $\theta = \frac{\pi}{4}$

$$x = a\sin^3\frac{\pi}{4} = a\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{a}{2\sqrt{2}}$$

and

$$y = a\cos^3\frac{\pi}{4} = a\left(\frac{1}{\sqrt{2}}\right)^3 = \frac{a}{2\sqrt{2}}$$

$\Rightarrow P\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$ is a point on (i) corresponding to

$$\theta = \frac{\pi}{4}.$$

Differentiating (i) w.r.t. θ, we get

$$\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$$

and

$$\frac{dy}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = -\cot\theta$$

$\therefore$ Slope of the tangent at

$$\theta = \frac{\pi}{4} \text{ is } -1$$

and slope of the normal at

$$\theta = \frac{\pi}{4} \text{ is } 1.$$

Hence equation of the tangent at P is

$$y - \frac{a}{2\sqrt{2}} = -1.\left(x - \frac{a}{2\sqrt{2}}\right)$$

$$\therefore \quad x + y = \frac{a}{\sqrt{2}}$$

and equation of the normal at P is

$$y - \frac{a}{2\sqrt{2}} = 1.\left(x - \frac{a}{2\sqrt{2}}\right)$$

$$\therefore \quad y = x$$ **Ans.**

17. Evaluate : $\int \frac{\sin^6 x + \cos^6 x}{\sin^2 x . \cos^2 x} dx$. **[4]**

Solution : Let, $I = \int \frac{\sin^6 x + \cos^6 x}{\sin^2 x \cos^2 x} dx$

Here Numerator $= \sin^6 x + \cos^6 x$

$$= (\sin^2 x + \cos^2 x)(\sin^4 x + \cos^4 x - \sin^2 x\cos^2 x)$$

$$[\because a^3 + b^3 = (a+b)(a^2 + b^2 - ab)]$$

$$= 1. [(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x - \sin^2 x \cos^2 x]$$

$$= 1 - 3\sin^2 x\cos^2 x$$

$$\therefore I = \int \frac{1 - 3\sin^2 x\cos^2 x}{\sin^2 x\cos^2 x} dx$$

$$= \int \frac{1}{\sin^2 x\cos^2 x} dx - 3\int 1.dx$$

Multiplying and dividing denominator by $\cos^2 x$ in first Integral

$$= \int \frac{\sec^4 x}{\tan^2 x} dx - 3x + C$$

$$= \int \frac{\sec^2 x . \sec^2 x}{\tan^2 x} dx - 3x + C$$

$$= \int \frac{(1 + \tan^2 x)\sec^2 x}{\tan^2 x} dx - 3x + C$$

Putting $\tan x = t$

$\Rightarrow \sec^2 x\, dx = dt$

$$\Rightarrow \quad I = \int \frac{(t^2 + 1)}{t^2} dt - 3x + C$$

$$= \int\left[1+\frac{1}{t^2}\right]dt - 3x + C$$

$$= t - \frac{1}{t} - 3x + C$$

$$= \tan x - \cot x - 3x + C.$$

OR

Evaluate : $\int (x-3)\sqrt{x^2+3x-18}\,dx.$

Solution : Let $I = \int (x-3)\sqrt{x^2+3x-18}\,dx$

$$= \int\left[\frac{1}{2}(2x+3) - \frac{9}{2}\right]\sqrt{x^2+3x-18}\,dx$$

$$= \frac{1}{2}\int (2x+3)\sqrt{x^2+3x-18}\,dx - \frac{9}{2}\int \sqrt{x^2+3x-18}\,dx$$

Putting, $x^2 + 3x - 18 = t$ in the first integral

$\Rightarrow \quad (2x+3)\,dx = dt$

$$\therefore \quad I = \frac{1}{2}\int \sqrt{t}\,dt - \frac{9}{2}\int \sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\,dx$$

$$= \frac{1}{2}\frac{t^{3/2}}{3/2} - \frac{9}{2}\left[\frac{1}{2}\left(x+\frac{3}{2}\right)\sqrt{\left(x+\frac{3}{2}\right)^2 - \left(\frac{9}{2}\right)^2}\right.$$

$$\left. - \frac{(9/2)^2}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}\right|\right] + C$$

$$\left[\because \int \sqrt{x^2-a^2}\,dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x+\sqrt{x^2-a^2}\right| + C\right]$$

$$= \frac{1}{3}(x^2+3x-18)^{3/2} - \frac{9}{4}\left[\left(x+\frac{3}{2}\right)\sqrt{x^2+3x-18} - \frac{81}{4}\log\left|x+\frac{3}{2}+\sqrt{x^2+3x-18}\right|\right] + C \quad \textbf{Ans.}$$

18. Find the particular solution of the differential equation $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$, **given that** $y = 1$ **when** $x = 0$. **[4]**

Solution : The given differential equation is

$$e^x\sqrt{1-y^2}\,dx + \frac{y}{x}dy = 0$$

Separating variables :

$$\int xe^x dx + \int \frac{y}{\sqrt{1-y^2}}dy = 0$$

On integrating, we get

$$xe^x dx + \frac{y}{\sqrt{1-y^2}}dy = C$$

Putting, $\quad 1 - y^2 = t$

$\Rightarrow \quad -2y.dy = dt$

$\Rightarrow \quad y.dy = -\frac{dt}{2}$

$\Rightarrow \quad x\int e^x dx - \int 1.e^x dx - \frac{1}{2}\int \frac{1}{\sqrt{t}}dt = C$

$\Rightarrow \quad xe^x - e^x - \frac{1}{2}\left[\frac{t^{1/2}}{\left(\frac{1}{2}\right)}\right] = C$

$\Rightarrow \quad e^x(x-1) - t^{1/2} = C$

$\Rightarrow \quad e^x(x-1) - \sqrt{1-y^2} = C \quad [\because t = 1 - y^2]$

Putting $\quad y = 1$

and $\quad x = 0$

$\Rightarrow \quad e^0(0-1) - \sqrt{1-1} = C$

$\Rightarrow \quad C = -1$

$\therefore \quad e^x(x-1) - \sqrt{1-y^2} = -1$ **Ans.**

19. Solve the following differential equation :

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1}$$ **[4]**

Solution : The given differential equation is

$$(x^2-1)\frac{dy}{dx} + 2xy = \frac{2}{x^2-1} \quad ...(i)$$

$$\Rightarrow \quad \frac{dy}{dx} + \frac{2x}{x^2-1}.y = \frac{2}{(x^2-1)^2} \quad ...(ii)$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $\quad P = \frac{2x}{x^2-1},$

and $\quad Q = \frac{2}{(x^2-1)^2}$

Now, $\quad \int P\,dx = \int \frac{2x}{x^2-1}dx = \log(x^2-1)$

$\therefore \quad I.F = e^{\int P dx} = e^{\log(x^2-1)} = x^2 - 1$

$\therefore$ The solution is

$$y.(\text{I.F.}) = \int Q(\text{I.F.})\,dx + C$$

$$\Rightarrow \quad y(x^2-1) = \int \frac{2}{(x^2-1)^2}\cdot(x^2-1)\,dx + C$$

$$= 2\int \frac{dx}{x^2-1} + C$$

$$= 2.\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$$

$\therefore$ The solution of the given differential equation is

$$y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C. \qquad \textbf{Ans.}$$

20. Prove that, for any three vectors : $\vec{a}, \vec{b}, \vec{c}$

$[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 2[\vec{a}, \vec{b}, \vec{c}]$ **[4]**

Solution : L. H. S. $= [\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}]$

$$= (\vec{a}+\vec{b}).[(\vec{b}+\vec{c})\times(\vec{c}+\vec{a})]$$

$$= (\vec{a}+\vec{b}).[\vec{b}\times\vec{c}+\vec{b}\times\vec{a}+\vec{c}\times\vec{c}+\vec{c}\times\vec{a}]$$

$$= (\vec{a}+\vec{b}).[\vec{b}\times\vec{c}-\vec{a}\times\vec{b}+\vec{0}+\vec{c}\times\vec{a}]$$

$$(\because \vec{c}\times\vec{c} = 0)$$

$$= \vec{a}.(\vec{b}\times\vec{c}) - \vec{a}.(\vec{a}\times\vec{b}) + \vec{a}.(\vec{c}\times\vec{a})$$
$$+ \vec{b}.(\vec{b}\times\vec{c}) - \vec{b}.(\vec{a}\times\vec{b}) + \vec{b}.(\vec{c}\times\vec{a})$$

$$= \vec{a}.(\vec{b}\times\vec{c}) - 0 + 0 + 0 - 0 + \vec{b}.(\vec{c}\times\vec{a})$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{b}, \vec{c}, \vec{a}]$$

$$\left[\begin{array}{l}\because \vec{a}.(\vec{b}\times\vec{c}) = [\vec{a}, \vec{b}, \vec{c}] \\ \text{and } \vec{b}.(\vec{c}\times\vec{a}) = [\vec{b}, \vec{c}, \vec{a}]\end{array}\right]$$

$$= [\vec{a}, \vec{b}, \vec{c}] + [\vec{a}, \vec{b}, \vec{c}]$$

$= 2[\vec{a}, \vec{b}, \vec{c}] = $ R.H.S. **Hence Proved.**

OR

Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c} = \vec{0}$ and $|\vec{a}| = 3, |\vec{b}| = 5$ and $|\vec{c}| = 7$. Find the angle between $\vec{a}$ and $\vec{b}$.

Solution : Given, $\quad |\vec{a}| = 3, |\vec{b}| = 5, |\vec{c}| = 7$

and $\quad \vec{a}+\vec{b}+\vec{c} = \vec{0}$

$$\Rightarrow \quad \vec{a}+\vec{b} = -\vec{c}$$

$$\Rightarrow \quad |\vec{a}+\vec{b}|^2 = |-\vec{c}|^2 = 7^2$$

$$\Rightarrow \quad (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 49$$

$$\Rightarrow \quad \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{b}.\vec{a}+\vec{b}.\vec{b} = 49$$

$$\Rightarrow \quad |\vec{a}|^2 + 2\vec{a}.\vec{b} + |\vec{b}|^2 = 49$$

$$\Rightarrow |\vec{a}|^2 + 2|\vec{a}||\vec{b}|\cos\theta + |\vec{b}|^2 = 49,$$

where θ is the angle between $\vec{a}$ and $\vec{b}$

$$\Rightarrow \quad 3^2 + 2\times3\times5\cos\theta + 5^2 = 49$$

$$\Rightarrow \quad 30\cos\theta = 15$$

$$\Rightarrow \quad \cos\theta = \frac{15}{30} = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \cos^{-1}(1/2)$$

$$\therefore \quad \theta = 60^\circ \qquad \textbf{Ans.}$$

21. Show that the lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect. Also find their point of intersection. **[4]**

Solution : Let $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} = r$ ...(i)

$\therefore$ Point (i) is $(3r-1, 5r-3, 7r-5)$.

And let $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5} = k$ ...(ii)

$\therefore$ Point (ii) is $(k+2, 3k+4, 5k+6)$

For lines (i) and (ii) to intersect, we get

$$3r-1 = k+2 \Rightarrow 3r-k = 3$$
$$5r-3 = 3k+4 \Rightarrow 5r-3k = 7$$
$$7r-5 = 5k+6 \Rightarrow 7r-5k = 11$$

Solving these equations, we get

$$r = \frac{1}{2};\ k = -\frac{3}{2}$$

$\therefore$ Lines (i) and (ii) intersect and their point of intersection is $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$. **Ans.**

22. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls ? Given that

(i) the youngest is a girl.

(ii) atleast one is a girl. **[4]**

Solution : The sample space

$$S = \{B_1B_2, B_1G_2, G_1G_2, G_1B_2\}$$

$$\Rightarrow \quad n(S) = 4$$

Let A be the event that both children are girls, B be the event that the youngest child is a girl and C be the event that atleast one of the children is a girl. Then

$$A = \{G_1G_2\}$$

$$\Rightarrow \quad n(A) = 1,$$

$$B = \{G_1G_2, B_1G_2\}$$

$$\Rightarrow \quad n(B) = 2,$$

and $$C = \{B_1G_2, G_1G_2, G_1B_2\}$$

$$\Rightarrow \quad n(C) = 3$$

$$\Rightarrow \quad A \cap B = \{G_1 G_2\}$$

$$\Rightarrow \quad n(A \cap B) = 1$$

and $$(A \cap C) = \{G_1 G_2\}$$

$$\Rightarrow \quad n(A \cap C) = 1$$

(i) The required probability

$$= P(A/B)$$

$$= \frac{P(A \cap B)}{P(B)} = \frac{\frac{n(A \cap B)}{n(S)}}{\frac{n(B)}{n(S)}}$$

$$= \frac{1/4}{2/4} = \frac{1}{2}$$

(ii) The required probability

$$= P(A/C)$$

$$= \frac{P(A \cap C)}{P(C)} = \frac{\frac{n(A \cap C)}{n(S)}}{\frac{n(C)}{n(S)}}$$

$$= \frac{1/4}{3/4} = \frac{1}{3} \qquad \text{Ans.}$$

SECTION — C

23. Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school P wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to its 3, 2 and 1 students with a total award money of ₹ 1,000. School Q wants to spend ₹ 1,500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of awards for one prize on each value is ₹ 600, using matrices, find the award money for each value.

Apart from the above three values, suggest one more value for awards. [6]

Solution : The awards for Discipline, Politeness and Punctuality is ₹ x, ₹ y and ₹ z respectively.

According to question,

$$3x + 2y + z = 1,000$$

$$4x + y + 3z = 1,500$$

$$x + y + z = 600$$

The given equation can be written in matrix form,

$$AX = B \qquad ...(i)$$

$$\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1,000 \\ 1,500 \\ 600 \end{bmatrix}$$

Here $$A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

And $$B = \begin{bmatrix} 1,000 \\ 1,500 \\ 600 \end{bmatrix}$$

Now, $$|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 3(1-3) - 2(4-3) + 1(4-1)$$

$$= -6 - 2 + 3$$

$$= -5 \neq 0$$

$\Rightarrow$ A^{-1} exists.

For adj A,

$$A_{11} = (1-3) = -2,$$
$$A_{12} = -(4-3) = -1,$$
$$A_{13} = (4-1) = 3$$
$$A_{21} = -(2-1) = -1,$$
$$A_{22} = (3-1) = 2,$$
$$A_{23} = -(3-2) = -1$$
$$A_{31} = (6-1) = 5,$$
$$A_{32} = -(9-4) = -5,$$
$$A_{33} = (3-8) = -5$$

$$\therefore \text{adj } A = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}^T = \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{-1}{5} \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$$

From (1), $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix} \begin{bmatrix} 1,000 \\ 1,500 \\ 600 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -2,00 & -1,500 & +3,000 \\ -1,000 & +3,000 & -3,000 \\ 3,000 & -1,500 & -3,000 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -500 \\ -1,000 \\ -1,500 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

$\Rightarrow \quad x =$ ₹ 100;

$y =$ ₹ 200

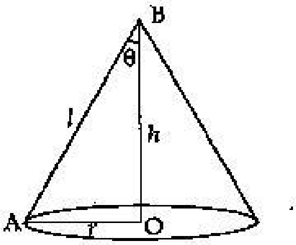
and $\quad z =$ ₹ 300.

A part from the three values, Discipline, Politeness and Punctuality, another value for award, should be Hard Work. **Ans.**

24. Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$. **[6]**

Solution : Let θ be the semi-vertical angle of a cone, *h* its height, *r* base radius and slant height:

$$l = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \quad \text{(Given)}$$



Then from ΔOAB,

$$r = l\sin\theta, h = l\cos\theta$$

Let V be the volume of the cone.

$$\text{Now,} \quad V = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi l^2 \sin^2\theta\, l\cos\theta$$

$$\therefore \quad V = \frac{1}{3}\pi l^3 \sin^2\theta.\cos\theta$$

Differentiating w.r. t. θ, we get

$$\frac{dV}{d\theta} = \frac{1}{3}\pi l^3 \,(2\sin\theta.\cos\theta.\cos\theta - \sin^2\theta.\sin\theta)$$

$$= \frac{1}{3}\pi l^3 \sin\theta(2\cos^2\theta - \sin^2\theta)$$

And again differentiating, we get

$$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi l^3 \;[\cos\theta\,(2\cos^2\theta - \sin^2\theta) + \sin\theta\,(-4\cos\theta\sin\theta - 2\sin\theta\cos\theta)]$$

For maxima or minima,

$$\frac{dV}{d\theta} = 0$$

$$\Rightarrow \quad \sin\theta\,(2\cos^2\theta - \sin^2\theta) = 0$$

$$\Rightarrow \quad \sin\theta = 0 \text{ or } 2\cos^2\theta - \sin^2\theta = 0$$

[$\because$ θ = 0 not possible]

$$\Rightarrow \quad 2\cos^2\theta - (1 - \cos^2\theta) = 0$$

$$\Rightarrow \quad 2\cos^2\theta - 1 + \cos^2\theta = 0$$

$$\Rightarrow \quad \cos^2\theta = \frac{1}{3}$$

$$\Rightarrow \quad \cos^2\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\text{For} \quad \cos\theta = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \sin\theta = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\therefore \quad \frac{d^2V}{d\theta^2} = \frac{1}{3}\pi l^3\left[\frac{1}{\sqrt{3}}\left(2.\frac{1}{3} - \frac{2}{3}\right) + \frac{\sqrt{2}}{\sqrt{3}}\left(-\frac{4\times 1}{\sqrt{3}}\times\frac{\sqrt{2}}{\sqrt{3}} - \frac{2\sqrt{2}}{\sqrt{3}}\,\frac{1}{\sqrt{3}}\right)\right]$$

$$= \frac{1}{3}\pi l^3\left[\frac{1}{\sqrt{3}}(0) + \frac{\sqrt{2}}{\sqrt{3}}\left(-\frac{4\sqrt{2}}{3} - \frac{2\sqrt{2}}{3}\right)\right]$$

$$= \frac{1}{3}\pi l^3\left[\frac{\sqrt{2}}{\sqrt{3}}\times\left(-\frac{6\sqrt{2}}{3}\right)\right]$$

$$= \frac{1}{3}\pi l^3\left(-\frac{4}{\sqrt{3}}\right) < 0$$

$\therefore$ V is maximum for $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Hence Proved.

25. Evaluate : $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}$. **[6]**

Solution :

$$\text{Let} \quad I = \int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\cot x}}$$

$$= \int_{\pi/6}^{\pi/3}\frac{dx}{1+\frac{\sqrt{\cos x}}{\sqrt{\sin x}}}$$

$$= \int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx \quad \text{...(i)}$$

$$I = \int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2-x)}+\sqrt{\cos(\pi/2-x)}}dx$$

$$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx \text{ and } a+b = \frac{\pi}{6}+\frac{\pi}{3} = \frac{\pi}{2}\right]$$

$$\therefore \quad I = \int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx \quad \text{...(ii)}$$

Adding (i) and (ii), we get

$\therefore \quad I + I = \int_{\pi/6}^{\pi/3}\left[\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx\right]$

$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} 1.dx$

$2I = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

$\therefore \quad I = \frac{\pi}{12}.$ **Ans.**

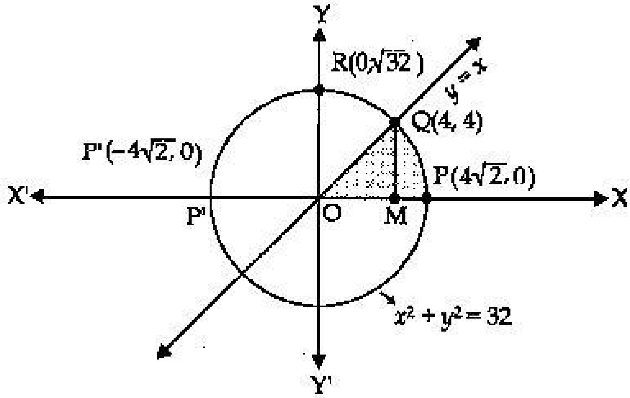
26. Find the area of the region in the first quadrant enclosed by the x-axis, the line $y = x$ and circle $x^2 + y^2 = 32$. **[6]**

Solution : Equation of the line and the circle are:

$y = x$ ...(i)

and $x^2 + y^2 = 32$ ...(ii)

$\Rightarrow \quad y = \sqrt{32 - x^2}$



Circle (ii) meets x-axis at

$P(4\sqrt{2}, 0)$ and $P'(-4\sqrt{2}, 0)$.

Also (ii) meets (i) at Q in the first quadrant and Q has co-ordinates (4, 4).

The required area in first quadrant = Area OPQ

= Area under line + Area under circle

$= \int_0^4 y_1 dx + \int_4^{4\sqrt{2}} y_2 dx$

$= \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$

$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x\sqrt{32-x^2}}{2} + \frac{32}{2}\sin^{-1}\left(\frac{x}{4\sqrt{2}}\right)\right]_4^{4\sqrt{2}}$

$= 8 + \left[0 + 16\sin^{-1}(1) - \frac{4.4}{2} - 16\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right]$

$= 16\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = 16 \times \frac{\pi}{4} = 4\pi$ sq. units. **Ans.**

27. Find the distance between the point (7, 2, 4) and the plane determined by the points A (2, 5, −3), B (−2, −3, 5) and C (5, 3, −3). **[6]**

Solution : The plane passing through A (2, 5, −3) is

$a(x-2) + b(y-5) + c(z+3) = 0$ ...(i)

It passes through B (−2, −3, 5) and C (5, 3, −3);

So $\quad -4a - 8b + 8c = 0$ ...(ii)

$3a - 2b + 0c = 0$ ...(iii)

Solving equation (ii) and (iii), we get

$\frac{a}{0+16} = \frac{b}{24-0} = \frac{c}{8+24}$

$\Rightarrow \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{4}$ ...(iv)

From (i) and (iv), the equation of the required plane is

$2(x-2) + 3(y-5) + 4(z+3) = 0$

$\Rightarrow \quad 2x + 3y + 4z - 7 = 0$

$\therefore$ Distance of the point (7, 2, 4) from it

$= \left|\frac{2\times7+3\times2+4\times4-7}{\sqrt{2^2+3^2+4^2}}\right|$

$= \left|\frac{14+6+16-7}{\sqrt{4+9+16}}\right|$

$= \frac{29}{\sqrt{29}} = \sqrt{29}.$ **Ans.**

OR

Find the distance of the point (− 1, − 5, − 10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$.

Solution : The given plane is

$\vec{r}.(\hat{i} - \hat{j} + \hat{k}) = 5$

$\Rightarrow \quad x - y + z = 5$ ...(i)

The given line is

$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$

$\Rightarrow \quad \frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} = \lambda$ ...(ii)

$\therefore$ Point (ii) is $(3\lambda + 2, 4\lambda - 1, 2\lambda + 2)$.

Let it lie on (i), so

$3\lambda + 2 - (4\lambda - 1) + 2\lambda + 2 = 5$

$\Rightarrow \quad \lambda = 0$

$\therefore$ The point of intersection of (i) and (ii) is (2, − 1, 2). Its distance from (− 1, − 5, − 10)

$= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)^2}$

$= \sqrt{9+16+144} = \sqrt{169} = 13.$ **Ans.**

28. A dealer in rural area wishes to purchase a number of sewing machines. He has only ₹ 5,760 to invest and has space for at most 20 items for storage. An electronic sewing machine cost him ₹ 360 and a manually operated sewing machine ₹ 240. He can sell an electronic sewing machine at a profit of ₹ 22 and a manually operated sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit ? Make it as a LPP and solve it graphically. [6]

Solution : Let the dealer buy x electronic and y manually operated sewing machines. The LPP is Maximize

$$Z = 22x + 18y$$

Subject to constraints :

$$x + y \leq 20$$

$$360x + 240y \leq 5,760$$

$$\Rightarrow \quad 3x + 2y \leq 48$$

and $\quad x \geq 0, y \geq 0$

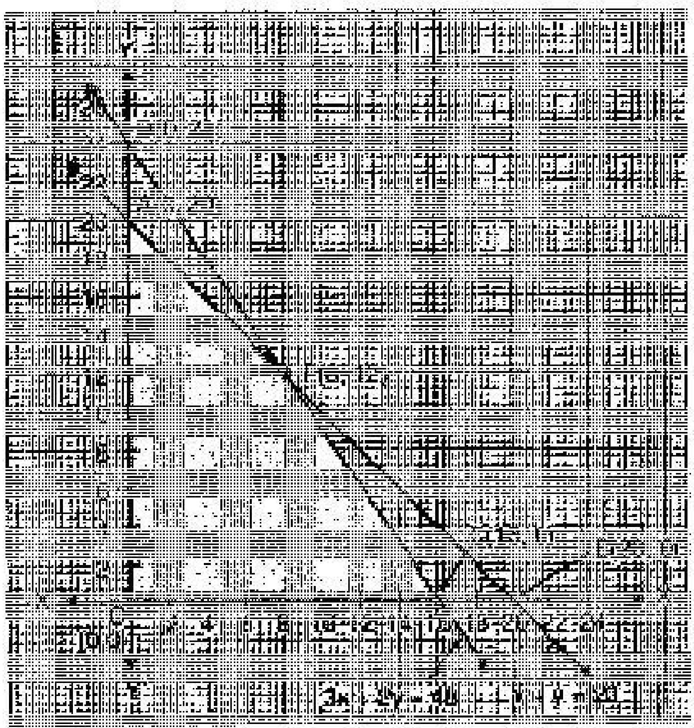
First we draw the lines AB and CD whose equations are

$$x + y = 20$$

	A	B
x	20	0
y	0	20

and $\quad 3x + 2y = 48$

	C	D
x	16	0
y	0	24



$\therefore$ The feasible region is OCPAO which is shaded in the figure.

The vertices of the feasible region are O (0, 0), C (16, 0), A (0, 20).

P is the point of intersection of the lines :

$$x + y = 20 \text{ and } 3x + 2y = 48.$$

Solving these equations, we get point P (8, 12).

The value of objective function $Z = 22x + 18y$ at these vertices are as follows :

Corner points	$Z = 22x + 18y$
At O (0, 0)	Z = 0
At C (16, 0)	Z = 352
At P (8, 12)	Z = 392 maximum
At A (0, 20)	Z = 360

$\therefore$ The maximum profit is ₹ 392 when 8 electronic and 12 manually operated machines are purchased.

29. A card from a pack of 52 playing cards is lost. From the remaining cards of the pack three cards are drawn at random (without replacement) and are found to be all spades. Find the probability of the lost card being a spade. [6]

Solution : Let E_1, E_2, E_3, E_4 and A be the events defined as below :

E_1 = the missing card is a heart card
E_2 = the missing card is a spade card
E_3 = the missing card is a club card
E_4 = the missing card is a diamond card

A = drawing three spades cards from the remaining cards.

$$P(E_1) = \frac{13}{52} = \frac{1}{4}$$

$$P(E_2) = \frac{13}{52} = \frac{1}{4}$$

$$P(E_3) = \frac{13}{52} = \frac{1}{4}$$

$$P(E_4) = \frac{13}{52} = \frac{1}{4}$$

$$P(A/E_1) = \frac{^{13}C_3}{^{51}C_3}$$

$$P(A/E_2) = \frac{^{12}C_3}{^{51}C_3}$$

$$P(A/E_3) = \frac{^{13}C_3}{^{51}C_3}$$

$$P(A/E_4) = \frac{^{13}C_3}{^{51}C_3}$$

By Bayes' theorem,

Required Probability = $P(E_2/A)$

$$= \frac{P(E_2)P(A/E_2)}{P(A/E_1).P(E_1) + P(A/E_2).P(E_2) + P(A/E_3).P(E_3) + P(A/E_4).P(E_4)}$$

$$= \frac{\frac{1}{4} \times \frac{^{12}C_3}{^{51}C_3}}{\frac{^{13}C_3}{^{51}C_3} \times \frac{1}{4} + \frac{^{12}C_3}{^{51}C_{13}} \times \frac{1}{4} + \frac{^{13}C_3}{^{51}C_3} \times \frac{1}{4} + \frac{^{13}C_3}{^{51}C_3} \times \frac{1}{4}}$$

$$= \frac{^{12}C_3}{^{12}C_3 + 3 \times {^{13}C_3}}$$

$$= \frac{12 \cdot 11 \cdot 10}{12 \cdot 11 \cdot 10 + 3 \cdot 13 \cdot 12 \cdot 11}$$

$$= \frac{10}{10 + 39} = \frac{10}{49}$$ **Ans.**

OR

From a lot of 15 bulbs which include 5 defectives, a sample of 4 bulbs is drawn one by one with replacement. Find the probability distribution of number of defective bulbs. Hence find the mean of the distribution.

Solution : Let D be the event of drawing a defective bulb and X denote the variable showing the number of defective bulbs in 4 draws. Then

$$P(D) = \frac{5}{15} = \frac{1}{3}$$

$$\Rightarrow \quad P(\overline{D}) = 1 - \frac{1}{3} = \frac{2}{3}$$

The drawn bulb is replaced.

Hence X takes values 0, 1, 2, 3 and 4.

$\therefore$ P(X = 0) = P (Getting no defective bulb)

$$= {^4C_0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$$

$$P(X = 1) = {^4C_1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3 = \frac{32}{81}$$

$$P(X = 2) = {^4C_2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{24}{81}$$

$$P(X = 3) = {^4C_3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right) = \frac{8}{81}$$

$$P(X = 4) = {^4C_4}\left(\frac{1}{3}\right)^4 = \frac{1}{81}$$

$\therefore$ The probability distribution is

X	0	1	2	3	4
P(X = x)	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$

Mean of the distribution = $\Sigma[P_i X_i]$

$$= \frac{16}{81} \times 0 + \frac{32}{81} \times 1 + \frac{24}{81} \times 2 + \frac{8}{81} \times 3 + \frac{1}{81} \times 4$$

$$= \frac{1}{81}(0 + 32 + 48 + 24 + 4) = \frac{108}{81} = \frac{4}{3}$$ **Ans.**

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — A

9. Evaluate : $\int \cos^{-1}(\sin x)\,dx$. **[1]**

Solution : $\int \cos^{-1}(\sin x)\,dx$

$$= \int \cos^{-1}\left[\cos\left(\frac{\pi}{2}-x\right)\right]dx$$

$$\left(\because \cos\left(\frac{\pi}{2}-x\right)=\sin x\right)$$

$$= \int\left(\frac{\pi}{2}-x\right)dx$$

$$= \frac{\pi}{2}x-\frac{x^2}{2}+C. \quad \textbf{Ans.}$$

10. If vectors $\vec{a}$ and $\vec{b}$ are such that, $|\vec{a}|=3$, $|\vec{b}|=\frac{2}{3}$ and $\vec{a}\times\vec{b}$ is a unit vector, then write the angle between $\vec{a}$ and $\vec{b}$. **[1]**

Solution : Given, $|\vec{a}| = 3, |\vec{b}| = \frac{2}{3}$

and $|\vec{a}\times\vec{b}| = 1$

We know that $|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta$

$$\Rightarrow \quad |\vec{a}||\vec{b}|\sin\theta = 1$$

$$\Rightarrow \quad 3.\frac{2}{3}\sin\theta = 1$$

$$\Rightarrow \quad \sin\theta = \frac{1}{2}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{2}\right)$$

$$\therefore \quad \theta = \frac{\pi}{6}$$

Hence the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$. **Ans.**

SECTION — B

19. Prove the following using properties of determinants :

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3 \quad [4]$$

Solution :

$$\text{L.H.S.} = \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

Taking $2(a+b+c)$ common from C_1, we get

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$, we get

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

Expanding along C_1, we get

$$2(a+b+c)\,[1.(b+c+a).(c+a+b)-0]$$

$$= 2(a+b+c)\,(a+b+c)^2$$

$$= 2(a+b+c)^3 = \text{R.H.S.} \quad \textbf{Hence Proved.}$$

20. Differentiate $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$ with respect to $\sin^{-1}(2x\sqrt{1-x^2})$. **[4]**

Solution : Let $y = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Put $x = \sin\theta$

$$\therefore \quad y = \tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$[\because 1-\sin^2\theta = \cos^2\theta]$$

$$= \tan^{-1}(\tan\theta)$$

$$\therefore \quad y = \theta$$

Differentiating w.r.t. to θ, we get

$$\frac{dy}{d\theta} = 1 \quad \ldots(i)$$

and let $t = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin\theta$

$\therefore \quad t = \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$

$= \sin^{-1}(2\sin\theta\cos\theta)$

$\therefore \quad t = 2\theta$

Differentiating w.r. to θ, we get

$$\frac{dt}{d\theta} = 2 \quad \text{...(ii)}$$

Now, $\frac{dy}{dt} = \frac{dy/d\theta}{dt/d\theta}$

$= \frac{1}{2}$. [Using (i) and (ii)] **Ans.**

21. Solve the following differential equation :

$\text{cosec}\, x \log y \frac{dy}{dx} + x^2y^2 = 0.$ [4]

Solution : The given differential equation is

$$\text{cosec}\, x \log y \frac{dy}{dx} + x^2y^2 = 0$$

Separating the variables, we get

$$\frac{\log y}{y^2}dy + \frac{x^2}{\text{cosec}\, x}dx = 0$$

On integrating, we get

$$\int \frac{\log y}{y^2}dy + \int x^2 \sin x\, dx = C$$

Put $\log y = t$

$\Rightarrow \quad \frac{1}{y}dy = dt$ and $y = e^t$

$\Rightarrow \quad \int t.e^{-t}dt + \int x^2 \sin x\, dx = C$

Integrate both by parts, we get

$$t.\frac{e^{-t}}{-1} - \int 1.\frac{e^{-t}}{-1}dt + x^2(-\cos x) - \int 2x(-\cos x)dx = C$$

$$\Rightarrow -te^{-t} - e^{-t} - x^2\cos x + 2x\sin x - 2\int 1.\sin x\, dx = C$$

$$\Rightarrow \quad \frac{-(1+t)}{e^t} - x^2\cos x + 2x\sin x - 2(-\cos x) = C$$

∴ The solution of the given differential equation is

$$-\left(\frac{1+\log y}{y}\right) - x^2\cos x + 2x\sin x + 2\cos x = C$$

Ans.

22. Show that the lines $\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$ are coplanar. [4]

Solution : The given lines are

$$\frac{5-x}{-4} = \frac{y-7}{4} = \frac{z+3}{-5}$$

and $$\frac{x-8}{7} = \frac{2y-8}{2} = \frac{z-5}{3}$$

$$\Rightarrow \quad \frac{x-5}{4} = \frac{y-7}{4} = \frac{z+3}{-5}$$

and $\frac{x-8}{7} = \frac{y-4}{1} = \frac{z-5}{3}$

We know that the given lines are coplanar. If

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Taking $\quad$ L. H. S. $= \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$

Here, $x_1 = 5, y_1 = 7, z_1 = -3, x_2 = 8, y_2 = 4, z_2 = 5, l_1 = 4, m_1 = 4, n_1 = -5, \quad l_2 = 7, m_2 = 1, n_2 = 3$

$$= \begin{vmatrix} 8-5 & 4-7 & 5-(-3) \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -3 & 8 \\ 4 & 4 & -5 \\ 7 & 1 & 3 \end{vmatrix}$$

$= 3(12+5) + 3(12+35) + 8(4-28)$

$= 51 + 141 - 192 = 0 = \text{R. H. S.}$

Hence, the given lines are coplanar.

Hence Proved.

SECTION — C

28. Evaluate : $\int_0^{\pi} \frac{x\tan x}{\sec x.\text{cosec}\, x}dx.$ [6]

Solution :

Let, $\quad I = \int_0^{\pi} \frac{x\tan x}{\sec x.\text{cosec}\, x}dx$

$$= \int_0^{\pi} \frac{x\sin x}{\cos x}\cdot\cos x.\sin x\, dx$$

$\therefore \quad I = \int_0^{\pi} x\sin^2 x\, dx \quad$ ...(i)

$I = \int_0^{\pi} (\pi - x)\sin^2(\pi - x)dx$

$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$\therefore \quad I = \int_0^{\pi} (\pi - x)\sin^2 x\, dx \quad ...(ii)$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi} [x\sin^2 x + (\pi - x)\sin^2 x\, dx]$$

$$= \int_0^{\pi} \sin^2 x(x + \pi - x)\, dx$$

$$= \pi \int_0^{\pi} \sin^2 x\, dx$$

$$= \pi \int_0^{\pi} \frac{1-\cos 2x}{2} dx$$

$$\left(\because \cos 2x = 1 - \sin^2 x\right)$$

$$= \frac{\pi}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi}$$

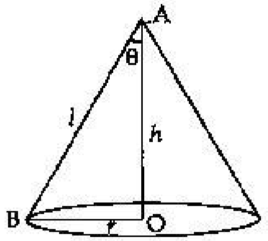
$$\Rightarrow \quad 2I = \frac{\pi}{2}[\pi - 0]$$

$$\Rightarrow \quad 2I = \frac{\pi^2}{2}$$

$$\therefore \quad I = \frac{\pi^2}{4} \qquad \textbf{Ans.}$$

29. Prove that the semi-vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. **[6]**

Solution : Let r, h, l, V and S be the base radius, height, slant height, volume (given) and curved surface of the cone respectively. Then



$$\therefore \quad \text{Slant height } l^2 = r^2 + h^2 \quad ...(i)$$

$$V = \frac{1}{3}\pi r^2 h \text{ (given)}$$

$$\Rightarrow \quad r^2 = \frac{3V}{\pi h} \quad ...(ii)$$

and $\quad S = \pi r l = \pi r\sqrt{r^2 + h^2}$

$$(\because l^2 = r^2 + h^2)$$

$$\Rightarrow \quad S^2 = \pi^2 r^2 (r^2 + h^2)$$

$$= \pi^2 . \frac{3V}{\pi h}\left(\frac{3V}{\pi h} + h^2\right)$$

[Using (ii)]

$$\therefore \quad S^2 = 3\pi V\left(\frac{3V}{\pi h^2} + h\right)$$

Fzor S to be least, S^2 is also least.

$$\therefore \quad \frac{dS^2}{dh} = 3\pi V.\left(\frac{-6V}{\pi h^3} + 1\right)$$

and $\quad \frac{d^2S^2}{dh^2} = 3\pi V\left(\frac{-6V}{\pi}\right)\frac{-3}{h^4}$

$$= \frac{54V^2}{h^4} \quad ...(iii)$$

For max. or min. S (and so S^2),

$$\frac{dS^2}{dh} = 0$$

$$\Rightarrow \quad 6V = \pi h^3$$

$$\Rightarrow \quad h = \left(\frac{6V}{\pi}\right)^{1/3} \quad ...(iv)$$

$\therefore$ From (iii),

$$\frac{d^2S^2}{dh^2} = \frac{54V^2}{h^4} > 0(+ve)$$

$\Rightarrow S^2$ and therefore S is least.

In right angled Δ AOB,

$$\cot\theta = \frac{h}{r}$$

$$= \frac{h}{\sqrt{3V/\pi h}} = \sqrt{\frac{\pi}{3V}}.h^{3/2}$$

[From (ii)]

$$\Rightarrow \quad \cot\theta = \sqrt{\frac{\pi}{3V}}.\sqrt{\frac{6V}{\pi}} = \sqrt{2} \quad \text{[From (iv)]}$$

$\therefore$ The semivertical angle,

$$\theta = \cot^{-1}(\sqrt{2}) \qquad \textbf{Ans.}$$

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Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION — A

9. Evaluate : $\int_0^{\pi/2} e^x(\sin x - \cos x)\,dx.$ [1]

Solution :

$$\text{Let } I = \int_0^{\pi/2} e^x(\sin x - \cos x)\,dx$$

$$= \int_0^{\pi/2} e^x \sin x\,dx - \int_0^{\pi/2} e^x \cos x\,dx$$

On integrating IInd integral by parts, we get

$$\int_0^{\pi/2} e^x \sin x\,dx - \left[\cos x \int_0^{\pi/2} e^x dx - \int_0^{\pi/2}\left[\frac{d}{dx}\cos x \int e^x dx\right]dx\right]$$

$$= \int_0^{\pi/2} e^x \sin x\,dx - \left[\{e^x \cos x\}_0^{\pi/2} - \int_0^{\pi/2} e^x(-\sin x)\,dx\right]$$

$$= \int_0^{\pi/2} e^x \sin x\,dx - [e^{\pi/2}.0 - e^0.1] - \int_0^{\pi/2} e^x \sin x\,dx$$

$$= -(0-1) = 1.$$ **Ans.**

10. Write a unit vector in the direction of the sum of the vectors $\vec{a} = 2\hat{i}+2\hat{j}-5\hat{k}$ and $\vec{b} = 2\hat{i}+\hat{j}-7\hat{k}$. [1]

Solution :

Given, $\vec{a} = 2\hat{i}+2\hat{j}-5\hat{k}$

$\vec{b} = 2\hat{i}+\hat{j}-7\hat{k}$

$\Rightarrow \vec{a}+\vec{b} = (2\hat{i}+2\hat{j}-5\hat{k})+(2\hat{i}+\hat{j}-7\hat{k})$

$= 4\hat{i}+3\hat{j}-12\hat{k}$

and $|\vec{a}+\vec{b}| = \sqrt{4^2+3^2+(-12)^2}$

$= \sqrt{16+9+144} = \sqrt{169} = 13$

The unit vector in the direction of $\vec{a}+\vec{b}$ is

$$\frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|} = \frac{4\hat{i}+3\hat{j}-12\hat{k}}{13}$$

$$= \frac{4}{13}\hat{i}+\frac{3}{13}\hat{j}-\frac{12}{13}\hat{k}.$$ **Ans.**

SECTION — B

19. Using properties of determinants, prove the following :

$$\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = 1+x^2+y^2+z^2$$ [4]

Solution : Taking L. H. S.

$$\text{Let } \Delta = \begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix}$$

Multiply C_1, C_2, C_3 by x, y, z respectively, we get

$$\Delta = \frac{1}{xyz}\begin{vmatrix} x(x^2+1) & xy^2 & xz^2 \\ x^2y & y(y^2+1) & yz^2 \\ x^2z & y^2z & (z^2+1)z \end{vmatrix}$$

Taking x, y, z common from R_1, R_2, R_3 respectively, we get

$$\Delta = \frac{1}{xyz}\cdot xyz\begin{vmatrix} x^2+1 & y^2 & z^2 \\ x^2 & y^2+1 & z^2 \\ x^2 & y^2 & z^2+1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} x^2+y^2+z^2+1 & y^2 & z^2 \\ x^2+y^2+z^2+1 & y^2+1 & z^2 \\ x^2+y^2+z^2+1 & y^2 & z^2+1 \end{vmatrix}$$

Taking $(x^2+y^2+z^2+1)$ common from C_1, we get

$$\Delta = (x^2+y^2+z^2+1)\begin{vmatrix} 1 & y^2 & z^2 \\ 1 & y^2+1 & z^2 \\ 1 & y^2 & z^2+1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$, we get

$$\Delta = (x^2+y^2+z^2+1)\begin{vmatrix} 1 & y^2 & z^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we get

$(x^2+y^2+z^2+1).\,1.\,(1-0)$

$= x^2+y^2+z^2+1 = \text{R. H. S.}$

Hence Proved.

20. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x \neq 0$. **[4]**

Solution : Let, $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

Putting $x = \tan\theta$,

$$\therefore \quad y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$$

$$\therefore \quad y = \frac{\theta}{2}$$

Differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = \frac{1}{2} \qquad \text{...(i)}$$

and let $t = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Putting $x = \tan\theta$

$$t = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \sin^{-1}\left(\frac{2\tan\theta}{\sec^2\theta}\right)$$

$(\because 1+\tan^2\theta = \sec^2\theta)$

$$= \sin^{-1}(2\sin\theta\cos\theta)$$

$$= \sin^{-1}(\sin 2\theta)$$

$$\therefore \quad t = 2\theta$$

Differentiating w.r.t. θ, we get

$$\frac{dt}{d\theta} = 2 \qquad \text{...(ii)}$$

Using (i) and (ii), we get

$$\frac{dy}{dt} = \frac{dy/d\theta}{dt/d\theta} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$$ **Ans.**

21. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$ given that $y = \frac{\pi}{2}$ when $x = 1$. **[4]**

Solution : The given differential equation is

$$\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y} \qquad \text{...(i)}$$

Separating the variables, we get

$(\sin y + y\cos y)\,dy = x\,(2\log x + 1)\,dx$

On integrating both sides, we get

$\int \sin y\,dy + \int y\cos y\,dy = \int x(2\log x+1)\,dx$

$\Rightarrow -\cos y + y.\sin y - \int 1.\sin y\,dy$

$$= \frac{x^2}{2}(2\log x+1) - \int \frac{x^2}{2}\cdot\frac{2}{x}dx + C$$

$\Rightarrow -\cos y + y\sin y - (-\cos y)$

$$= \frac{x^2}{2}(2\log x+1) - \frac{x^2}{2} + C$$

$\Rightarrow \quad y\sin y = x^2\log x + C$

Putting $y = \frac{\pi}{2}$ and $x = 1$, we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1^2.\log 1 + C$$

$$\Rightarrow \quad \frac{\pi}{2}.1 = 0 + C \ \Rightarrow C = \frac{\pi}{2}$$

$\therefore$ The particular solution of the given differential equation is

$$y\sin y = \frac{\pi}{2} + x^2\log x$$ **Ans.**

22. Show that lines $\vec{r} = (\hat{i}+\hat{j}-\hat{k}) + \lambda(3\hat{i}-\hat{j})$ and $\vec{r} = (4\hat{i}-\hat{k}) + \mu(2\hat{i}+3\hat{k})$ intersect. Also find their point of intersection. **[4]**

olution : The equation of given lines are

$$\vec{r} = (\hat{i}+\hat{j}-\hat{k}) + \lambda(3\hat{i}-\hat{j}) \qquad \text{...(i)}$$

$$\vec{r} = (4\hat{i}-\hat{k}) + \mu(2\hat{i}+3\hat{k}) \qquad \text{...(ii)}$$

For lines (i) and (ii) to be intersecting

$(\hat{i}+\hat{j}-\hat{k}) + \lambda(3\hat{i}-\hat{j}) = (4\hat{i}-\hat{k}) + \mu(2\hat{i}+3\hat{k})$

for some values of λ and μ.

$\Rightarrow (1+3\lambda)\hat{i} + (1-\lambda)\hat{j} - \hat{k} = (4+2\mu)\hat{i} + (3\mu-1)\hat{k}$

On equating coefficients,

$$1 + 3\lambda = 4 + 2\mu \qquad \text{...(iii)}$$

$$1 - \lambda = 0$$

$$-1 = 3\mu - 1$$

$\Rightarrow \qquad \lambda = 1$ and $\mu = 0$.

These values of λ and μ satisfy the equation (iii).

$\therefore$ Lines (i) and (ii) intersect at the point whose position vector is $4\hat{i} - \hat{k}$.

Thus, the coordinates of the point of intersection are (4, 0, – 1). **Ans.**

SECTION — C

28. Evaluate : $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$ **[6]**

Solution :

Let $\quad I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} \cdot dx \qquad$ ...(i)

$$\therefore \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \sin\left(\frac{\pi}{2} - x\right) \cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x \sin x}{\cos^4 x + \sin^4 x} dx$$

$$\left(\because \sin\left(\frac{\pi}{2} - x\right) = \cos x \text{ and } \cos\left(\frac{\pi}{2} - x\right) = \sin x\right)$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \sin x \cos x}{\sin^4 x + \cos^4 x} dx \qquad \text{...(ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\frac{\pi}{2} \sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

Dividing Nr and Dr by $\cos^4 x$, we get

$$I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\tan x \sec^2 x}{\tan^4 x + 1} dx$$

Putting $\qquad \tan^2 x = t$

$\Rightarrow 2 \tan x \sec^2 x \, dx = dt$

Also, $\qquad x = 0 \Rightarrow t = 0$

and $\qquad x = \frac{\pi}{2} \Rightarrow t = \infty$

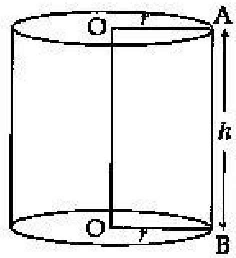
$$\therefore \quad I = \frac{\pi}{8} \int_0^{\infty} \frac{dt}{t^2 + 1} = \frac{\pi}{8} \left[\tan^{-1} t\right]_0^{\infty}$$

$$= \frac{\pi}{8} [\tan^{-1} \infty - \tan^{-1} 0]$$

$$= \frac{\pi}{8}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{16} \qquad \textbf{Ans.}$$

29. Of all the closed right circular cylindrical cans of volume 128 π cm^3, find the dimensions of the can which has minimum surface area. [6]

Solution : Let r cm be the base radius and h cm be the height of the closed cylindrical cans of given volume = 128 π cm^3.



Then volume, $\qquad V = \pi r^2 h = 128\pi$

$$\Rightarrow \qquad h = \frac{128}{r^2} \qquad \text{...(i)}$$

Also surface area, $\qquad S = 2\pi r h + 2\pi r^2$

$$= 2\pi\left(r \cdot \frac{128}{r^2} + r^2\right)$$

$$\Rightarrow \qquad S = 2\pi\left(r^2 + \frac{128}{r}\right)$$

Differentiating w.r.t. r, we get

$$\therefore \qquad \frac{dS}{dr} = 2\pi\left(2r - \frac{128}{r^2}\right)$$

$$\text{and} \qquad \frac{d^2S}{dr^2} = 2\pi\left(2 + \frac{128 \times 2}{r^3}\right)$$

For maxima or minima,

$$\frac{dS}{dr} = 0$$

$$\Rightarrow \qquad 2\pi\left(2r - \frac{128}{r^2}\right) = 0$$

$\Rightarrow \qquad 2r^3 = 128$

$\Rightarrow \qquad r^3 = 64$

$\Rightarrow \qquad r = 4$ cm

$$\text{and} \qquad \frac{d^2S}{dr^2} = 2\pi\left(2 + \frac{256}{64}\right)$$

$$= 12\pi > 0$$

$\Rightarrow$ S is minimum.

The dimensions of such a can are

$r = 4$ cm

$$\text{and} \qquad h = \frac{128}{r^2} = \frac{128}{4^2}$$

$$= \frac{128}{16} = 8 \text{ cm.} \qquad \textbf{Ans.}$$

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