Mathematics 2012 (Outside Delhi) Set I

Time allowed : 3 Hours **Maximum marks : 100**

SECTION — A

1. The binary operation $* : R \times R \to R$ is defined as $a * b = 2a + b$. Find $(2 * 3) * 4$.** [1]

2. Find the principal value of $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ [1]

Solution : $\tan^{-1}\sqrt{3} - \sec^{-1}(-2)$ ...(i)

We know that the range of principal value of $\tan^{-1}\theta$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$\therefore \quad \tan^{-1}\sqrt{3} = \tan^{-1}\left(\tan\frac{\pi}{3}\right) = \frac{\pi}{3}$

Now $\sec^{-1}(-2)$

We know that the range of principal value of $\sec^{-1}\theta$ is $[0, \pi] - \{\pi/2\}$

$\Rightarrow \quad \sec^{-1}(-2) = \pi - \sec^{-1}(2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$[\because \sec^{-1}(-x) = \pi - \sec^{-1} x]$

$\therefore$ From equation (i)

$\tan^{-1}\sqrt{3} - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = -\frac{\pi}{3}$ **Ans.**

3. Find the value of $x + y$ from the following equation:

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$ [1]

Solution : Given that,

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 2x & 10 \\ 14 & 2y-6 \end{bmatrix} + \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

$$\begin{bmatrix} 2x+3 & 6 \\ 15 & 2y-4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}$$

Equating values, we get x and y

$2x + 3 = 7$

$\Rightarrow \quad 2x = 7 - 3$

$\Rightarrow \quad 2x = 4$

$\Rightarrow \quad x = 2$

and $\quad 2y - 4 = 14$

$\Rightarrow \quad 2y = 14 + 4$

$\Rightarrow \quad 2y = 18$

$\Rightarrow \quad y = 9$

$\Rightarrow \quad x = 2, y = 9$

$\therefore \quad x + y = 2 + 9 = 11.$

Ans.

4. If $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$, then find $A^T - B^T$. [1]

Solution : $A^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$\Rightarrow \quad B^T = \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$\Rightarrow \quad A^T - B^T = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3+1 & 4-1 \\ -1-2 & 2-2 \\ 0-1 & 1-3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix}$$ **Ans.**

5. Let A be a square matrix of order 3×3. Write the value of $|2A|$, where $|A| = 4$. [1]

Solution : In a square matrix of order 3×3,

$|KA| = K^3|A|$

$\therefore \quad |2A| = 2^3|A|$

$= 8 \times 4 = 32.$ **Ans.**

6. Evaluate : $\int_0^2 \sqrt{4 - x^2}\,dx$. [1]

Solution : We know that,

$$\int\sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

**Answer is not given due to the change in present syllabus

Therefore,

$$\int_0^2 \sqrt{2^2-x^2}\,dx = \left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$=\left[\frac{2}{2}\sqrt{4-4}+2\sin^{-1}\frac{2}{2}\right]-[0+2\sin^{-1}0]$$

$$=2\sin^{-1}(1)-2\sin^{-1}(0)$$

$$=2\left(\frac{\pi}{2}\right)-0$$

$$=\pi.$$

Ans.

7. **Given** $\int e^x(\tan x+1)\sec x\,dx = e^x f(x)+c$. **Write** $f(x)$ **satisfying the above.** **[1]**

Solution : $\int e^x(\tan x+1)\sec x\,dx = e^x f(x)+c$

Taking L.H.S.

Let $I = \int e^x(\tan x+1)\sec x\,dx$

$$= \int e^x(\sec x\tan x+\sec x)\,dx$$

$$= e^x\sec x + c$$

$$\left[\because \int e^x\{f(x)+f'(x)\}dx = e^x f(x)+c\right]$$

On comparing L.H.S. and R.H.S.

we get, $f(x) = \sec x.$ **Ans.**

8. **Write the value of** $(\hat{i}\times\hat{j}).\hat{k}+\hat{i}.\hat{j}$ **[1]**

Solution : $(\hat{i}\times\hat{j}).\hat{k}+\hat{i}.\hat{j}$

$$= \hat{k}.\hat{k}+0 \qquad \left[\because(\hat{i}\times\hat{j})=\hat{k} \text{ and } \hat{k}.\hat{k}=1\right]$$

$$= 1.$$

Ans.

9. **Find the scalar components of the vector $\overrightarrow{AB}$ with initial point A (2, 1) and terminal point B (– 5, 7).** **[1]**

Solution : Position vector of A

$$\overrightarrow{OA} = 2\hat{i}+\hat{j}$$

Position vector of B

$$\overrightarrow{OB} = -5\hat{i}+7\hat{j}$$

$$\overrightarrow{AB} = (-5\hat{i}+7\hat{j})-(2\hat{i}+\hat{j})$$

$$= -7\hat{i}+6\hat{j}$$

Scalar components of vector $\overrightarrow{AB}$ are –7 and 6.

Ans.

10. **Find the distance of the plane $3x - 4y + 12z = 3$ from the origin.** **[1]**

Solution : We know that the distance of the point (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is

$$= \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|-3|}{\sqrt{3^2+(-4)^2+12^2}}$$

$$= \frac{|-3|}{\sqrt{169}}$$

$$= \frac{3}{13}$$

Ans.

SECTION — B

11. **Prove the following :**

$$\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)=\frac{6}{5\sqrt{13}}.$$ **[4]**

Solution : Taking L.H.S.

$$= \cos\left(\sin^{-1}\frac{3}{5}-\cot^{-1}\frac{3}{2}\right)$$

$$= \cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{2}{\sqrt{13}}\right)$$

$$\left(\because \cot^{-1}\frac{3}{2}=\sin^{-1}\frac{2}{\sqrt{13}}\right)$$

...(i)

We know that,

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right]$$

Therefore, equation (i) becomes,

$$= \cos\left[\sin^{-1}\left(\frac{3}{5}\sqrt{1-\frac{4}{13}}+\frac{2}{\sqrt{13}}\sqrt{1-\frac{9}{25}}\right)\right]$$

$$= \cos\left[\sin^{-1}\left(\frac{3}{5}\sqrt{\frac{9}{13}}+\frac{2}{\sqrt{13}}\sqrt{\frac{16}{25}}\right)\right]$$

$$= \cos\left[\sin^{-1}\left(\frac{3}{5}\cdot\frac{3}{\sqrt{13}}+\frac{2}{\sqrt{13}}\cdot\frac{4}{5}\right)\right]$$

$$= \cos\left[\sin^{-1}\left(\frac{9}{5\sqrt{13}}+\frac{8}{5\sqrt{13}}\right)\right]$$

$$= \cos\left[\sin^{-1}\left(\frac{17}{5\sqrt{13}}\right)\right]$$

$$= \cos\left[\cos^{-1}\left(\frac{6}{5\sqrt{13}}\right)\right]$$

$$\left[\because \sin^{-1}\left(\frac{17}{5\sqrt{13}}\right)=\cos^{-1}\left(\frac{6}{5\sqrt{13}}\right)\right]$$

$$= \frac{6}{5\sqrt{13}} = \text{R.H.S.}$$ **Hence Proved.**

12. Using properties of determinants, show that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$ **[4]**

Solution : Taking L.H.S. = $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2(b+c) & 2(a+c) & 2(a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= 2\begin{vmatrix} b+c & a+c & a+b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$= 2\begin{vmatrix} c & 0 & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

On Expanding along R_1

$= 2[c\{(c+a)(a+b)-bc\}-0+a\{bc-c(c+a)\}]$

$= 2[c\{ca+bc+a^2+ab-bc\}+a\{bc-c^2-ca\}]$

$= 2[c^2a+ca^2+abc+abc-c^2a-ca^2]$

$= 2[abc+abc]$

$= 4abc =$ R.H.S. **Hence Proved.**

13. Show that $f: N \to N$, given by

$$f(x) = \begin{cases} x+1, \text{if } x \text{ is odd} \\ x-1, \text{if } x \text{ is even} \end{cases}$$

is both one-one and onto. **[4]**

Solution : Let $x, y \in R$ such that $f(x) = f(y)$

$\therefore \quad f(x) = f(y)$

If x and y are odd, then

$f(x) = f(y)$

$\Rightarrow \quad x+1 = y+1$

$\Rightarrow \quad x = y$

If x and y are even, then

$f(x) = f(y)$

$\Rightarrow \quad x-1 = y-1$

$\Rightarrow \quad x = y$

If x is odd and y is even, then

$f(x) = x+1$ is even and $f(y) = y+1$ is odd.

$\therefore \quad x \neq y \Rightarrow f(x) \neq f(y)$

Similarly if x is even and y is odd, then

$x \neq y \Rightarrow f(x) \neq f(y)$

Hence, $f: N \to N$ is one-one

Also, $f(1) = 1+1 = 2$

$f(1) = 2$ $(\because 1 \text{ is odd})$

If x is odd number, then $\exists$ an even natural number, $x+1 \in N$ such that,

$f(x+1) = x+1-1$

$= x$

If x is even number, then there exist a odd natural number $x-1 \in N$ such that,

$f(x-1) = x-1+1$

$= x$

Hence for every $y \in N \; \exists \; x \in N$ such that $f(x) = y$, so f is onto.

Hence f is both one-one and onto.

Hence Proved.

OR

Consider the binary operations $*: R \times R \to R$ and $o: R \times R \to R$ defined as $a*b = |a-b|$ and $aob = a$ for all $a, b \subset R$. Show that '*' is commutative but not associative, 'o' is associative but not commutative.**

14. If $x = \sqrt{a^{\sin^{-1} t}}, y = \sqrt{a^{\cos^{-1} t}}$, show that $\frac{dy}{dx} = -\frac{y}{x}$ **[4]**

Solution : Given, $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$

Squaring both sides

$$x^2 = a^{\sin^{-1} t} \quad ...(i)$$

$$y^2 = a^{\cos^{-1} t} \quad ...(ii)$$

Differentiating both w.r.t. t

$$2x\frac{dx}{dt} = a^{\sin^{-1} t} \log a. \frac{1}{\sqrt{1-t^2}},$$

and $$2y\frac{dy}{dx} = a^{\cos^{-1} t} \log a. \frac{-1}{\sqrt{1-t^2}}$$

$$\Rightarrow \quad \frac{dx}{dt} = \frac{a^{\sin^{-1} t} \log a}{2x\sqrt{1-t^2}},$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{a^{\cos^{-1} t} \log a}{2y\sqrt{1-t^2}}$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= -\frac{a^{\cos^{-1} t} \log a / 2y\sqrt{1-t^2}}{a^{\sin^{-1} t} \log a / 2x\sqrt{1-t^2}}$$

**Answer is not given due to the change in present syllabus

$$= \frac{-xa^{\cos^{-1}t}}{ya^{\sin^{-1}t}}$$

$$= \frac{-x}{y} \cdot \frac{y^2}{x^2} \quad \text{[Using (i) \& (ii)]}$$

$$= \frac{-y}{x}$$ **Hence Proved.**

OR

Differentiate $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ with respect to x.

Solution : Let $y = \tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$

Put $x = \tan A \Rightarrow A = \tan^{-1} x$ ...(i)

$$y = \tan^{-1}\left[\frac{\sqrt{1+\tan^2 A}-1}{\tan A}\right]$$

$$y = \tan^{-1}\left[\frac{\sqrt{\sec^2 A}-1}{\tan A}\right]$$

$$= \tan^{-1}\left[\frac{\sec A - 1}{\tan A}\right]$$

$$= \tan^{-1}\left[\frac{1-\cos A}{\sin A}\right]$$

$$= \tan^{-1}\left[\frac{2\sin^2\left(\frac{A}{2}\right)}{2\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{A}{2}\right)\right]$$

$$\Rightarrow \quad y = \frac{A}{2}$$

Put the value of A from equation (i),

$$y = \frac{\tan^{-1} x}{2}$$

On differentiating w.r.t. x

$$\frac{dy}{dx} = \frac{1}{2(1+x^2)}.$$ **Ans.**

15. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, $0 < t < \frac{\pi}{2}$, find $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$. **[4]**

Solution : Given, $x = a(\cos t + t \sin t)$

On differentiating w.r.t. t, we get

$$\frac{dx}{dt} = a[-\sin t + t\cos t + \sin t]$$

$$\Rightarrow \quad \frac{dx}{dt} = a(t\cos t)$$

Again differentiating w.r.t. t, we get

$$\frac{d^2x}{dt^2} = a[\cos t - t\sin t] \quad \text{...(i)}$$

Given, $y = a(\sin t - t\cos t)$

On differentiating w.r.t. t, we get

$$\frac{dy}{dt} = a[\cos t + t\sin t - \cos t]$$

$$\Rightarrow \quad \frac{dy}{dt} = a(t\sin t)$$

Again differentiating w.r.t. t, we get

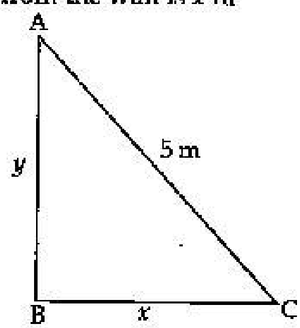
$$\frac{d^2y}{dt^2} = a[\sin t + t\cos t] \quad \text{...(ii)}$$

On dividing equation (ii) by equation (i), we get

$$\frac{d^2y}{dx^2} = \frac{t\cos t + \sin t}{\cos t - t\sin t}$$ **Ans.**

16. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground, away from the wall, at the rate of 2 cm/s. How fast is its height on the wall decreasing when the foot of the ladder is 4 m away from the wall ? **[4]**

Solution : Let AC = 5 m be the ladder and y be the height of the wall at which the ladder touches. Also, let the foot of the ladder be at C whose distance from the wall is x m



It is given that

$$\frac{dx}{dt} = 2 \text{ cm/sec} = \frac{2}{100} \text{ m/sec} \quad \text{...(i)}$$

As we know that Δ ABC is right angled triangle,

$$\therefore \quad x^2 + y^2 = 5^2 \quad \text{...(ii)}$$

On differentiating, we get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \quad 2x\frac{dx}{dt} = -2y\frac{dy}{dt}$$

$$\Rightarrow \quad x \times \frac{2}{100} = -y\frac{dy}{dt} \quad \text{[from (i)]}$$

$$\Rightarrow \quad \frac{dy}{dt} = \frac{-2x}{100y}$$

When $x = 4$ m from eq. (ii)

$$y^2 = 25 - x^2$$

$$\Rightarrow \quad y = \sqrt{25-16}$$

$$\Rightarrow \quad y = 3\text{ m}$$

Thus, $$\frac{dy}{dt} = \frac{-2\times 4}{100\times 3} = \frac{-2}{75}$$

[Negative sign shows that height of ladder on the wall is decreasing at the rate of $\frac{2}{75}$ m/s] **Ans.**

17. Evaluate : $\int_{-1}^{2} |x^3 - x|\, dx$. **[4]**

Solution : Let $f(x) = x^3 - x$

$$= x(x-1)(x+1)$$

Sign of $f(x)$ for different value of x will be different

$$I = \int_{-1}^{0} |x^3 - x|dx + \int_{0}^{1} |x^3 - x|dx + \int_{1}^{2} |x^3 - x|\,dx$$

$$= \int_{-1}^{0} (x^3 - x)dx - \int_{0}^{1}(x^3 - x)dx + \int_{1}^{2}(x^3 - x)dx$$

$$\left[\begin{array}{l} \because \text{At}(-1,0),\ x(x-1)(x+1) = (-\text{ve})(-\text{ve})(+\text{ve}) = +\text{ve} \\ \text{At}(0,1),\ x(x-1)(x+1) = (+\text{ve})(-\text{ve})(+\text{ve}) = -\text{ve} \\ \text{At}(1,2),\ x(x-1)(x+1) = (+\text{ve})(+\text{ve})(+\text{ve}) = +\text{ve} \end{array}\right]$$

$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{0}^{1} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2}$$

$$= 0 - \left[\frac{1}{4} - \frac{1}{2}\right] - \left[\left\{\frac{1}{4} - \frac{1}{2}\right\} - 0\right] + \left[\frac{16}{4} - \frac{4}{2}\right] - \left[\frac{1}{4} - \frac{1}{2}\right]$$

$$= \frac{11}{4}.$$ **Ans.**

OR

Evaluate : $\int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$.

Solution : Let $$I = \int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}dx \quad \text{...(i)}$$

Use the following $\int_{0}^{a} f(x)dx = \int_{0}^{a} f(a-x)dx$

Therefore,

$$I = \int_{0}^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$$

$$\Rightarrow \quad I = \int_{0}^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx \quad \text{...(ii)}$$

Adding equation (i) and (ii), we get

$$2I = \int_{0}^{\pi} \frac{x\sin x}{1+\cos^2 x}dx + \int_{0}^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$$

$$= \int_{0}^{\pi}\left[\frac{x\sin x + \pi\sin x - x\sin x}{1+\cos^2 x}\right]dx$$

$$\Rightarrow \quad 2I = \int_{0}^{\pi} \frac{\pi\sin x}{1+\cos^2 x}dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_{0}^{\pi} \frac{\sin x}{1+\cos^2 x}dx \quad \text{...(iii)}$$

Let $\cos x = t$

When $x = 0,\ t = \cos 0 = 1$

$x = \pi,\ t = \cos\pi = -1$

$$-\sin x = \frac{dt}{dx}$$

$$\sin x\, dx = -dt$$

Put the value of $\sin x\, dx$ in equation (iii), we get

$$I = \frac{\pi}{2}\int_{1}^{-1} \frac{-1}{1+t^2}dt$$

$$= \frac{-\pi}{2}\left[\tan^{-1} t\right]_{1}^{-1}$$

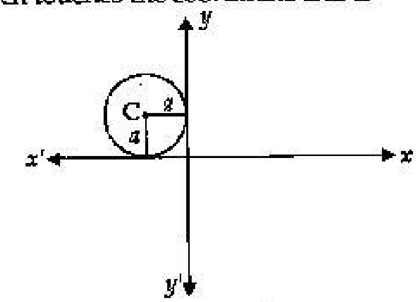
$$= \frac{-\pi}{2}\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$= \frac{-\pi}{2}\left[\frac{-\pi}{4} - \frac{\pi}{4}\right]$$

$$\Rightarrow \quad I = \frac{\pi^2}{4}.$$ **Ans.**

18. Form the differential equation of the family of circles in the second quadrant and touching the coordinate axes. **[4]**

Solution : The equation of circle in second quadrant which touches the coordinate axis is



Let a be the radius of circle

$\therefore$ centre $= (-a, a)$.

$$(x+a)^2 + (y-a)^2 = a^2 \quad \text{...(i)}$$

On differentiating w.r.t. x, we get

$$2(x+a)+2(y-a)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad x+a+y\frac{dy}{dx}-a\frac{dy}{dx} = 0$$

$$\Rightarrow \quad a\frac{dy}{dx}-a = x+y\frac{dy}{dx}$$

$$\Rightarrow \quad a\left(\frac{dy}{dx}-1\right) = x+y\frac{dy}{dx}$$

$$\Rightarrow \quad a = \frac{x+y\frac{dy}{dx}}{\frac{dy}{dx}-1}$$

$$\Rightarrow \quad a = \frac{x+yy_1}{y_1-1}, \text{ where } y_1 = \frac{dy}{dx}$$

Putting the value of a in eq. (i),

$$\left[x+\frac{x+yy_1}{y_1-1}\right]^2+\left[y-\frac{x+yy_1}{y_1-1}\right]^2 = \left[\frac{x+yy_1}{y_1-1}\right]^2$$

$$\Rightarrow y_1^2(x+y)^2+(x+y)^2=(x+yy_1)^2$$

$$\Rightarrow (x+y)^2[(y_1)^2+1]=(x+yy_1)^2.$$ **Ans.**

OR

Find the particular solution of the differential equation $x(x^2-1)\frac{dy}{dx}=1; y=0$ when $x=2$.

Solution : Given, $x(x^2-1)\frac{dy}{dx}=1; y=0, x=2$

$$dy = \frac{dx}{x(x^2-1)}$$

Integrating both sides, we get

$$\int dy = \int\frac{dx}{x(x^2-1)}$$

$$y = \int\frac{dx}{x(x^2-1)} \quad \text{...(i)}$$

Now $\quad \frac{1}{x(x^2-1)} = \frac{1}{x(x-1)(x+1)}$

Solving by partial fractions, we get

$$\frac{1}{x(x-1)(x+1)} = \frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1} \quad \text{...(ii)}$$

$A(x^2-1)+Bx(x+1)+Cx(x-1)=1$

Putting $x=0$,

$\Rightarrow A(-1)+B(0)+C(0)=1$

$\therefore \quad A=-1$

Putting $x=1$

$\Rightarrow \quad A(0)+B(2)+C(0)=1$

$\therefore \quad B=\frac{1}{2}$

Putting $x=-1$

$A(0)+B(0)+C(-1)(-2)=1$

$\therefore \quad C=\frac{1}{2}$

On substituting the values of A, B, C in equation (ii), we get

$$\frac{1}{x(x-1)(x+1)} = \frac{-1}{x}+\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$$

From eq. (i)

$$y = \int\frac{1}{x(x^2-1)}dx$$

$$= -\int\frac{dx}{x}+\frac{1}{2}\int\frac{dx}{x-1}+\frac{1}{2}\int\frac{dx}{x+1}$$

$$= -\log x+\frac{1}{2}\log(x-1)+\frac{1}{2}\log(x+1)+c$$

$$= \log\left(\frac{1}{x}\right)+\frac{1}{2}\log(x^2-1)+c$$

$$= \log\left[\frac{\sqrt{x^2-1}}{x}\right]+c$$

Now, putting $y=0, x=2$, we get

$$0 = \log\frac{\sqrt{3}}{2}+c$$

$$\Rightarrow \quad c = -\log\frac{\sqrt{3}}{2}$$

$\therefore$ The particular solution is

$$y = \log\left[\frac{\sqrt{x^2-1}}{x}\right]+\left(-\log\frac{\sqrt{3}}{2}\right)$$

$$= \frac{2}{2}\log\left[\frac{\sqrt{x^2-1}}{x}\right]+\frac{2}{2}\log\left[\frac{2}{\sqrt{3}}\right]$$

$$= \frac{1}{2}\log\left[\frac{x^2-1}{x^2}\right]+\frac{1}{2}\log\left[\frac{4}{3}\right]$$

$$= \frac{1}{2}\log\frac{4(x^2-1)}{3x^2}$$ **Ans.**

19. Solve the following differential equation : $(1+x^2)\,dy+2xy\,dx=\cot x\,dx; x\neq 0.$ **[4]**

Solution : $(1+x^2)\,dy+2xy\,dx=\cot x\,dx$

Dividing with $(1+x^2)\,dx$ on both sides, we get

$$\frac{dy}{dx}+\left(\frac{2x}{1+x^2}\right)y = \frac{\cot x}{1+x^2}$$

On comparing this equation with

$$\frac{dy}{dx}+Py = Q$$

Where $P=\frac{2x}{1+x^2}, Q=\frac{\cot x}{1+x^2}$

Now, $\quad \text{I.F.} = e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx}$

$$= e^{\log(1+x^2)} = 1 + x^2$$

Solution is

$$y\,(\text{I.F.}) = \int Q.(\text{I.F.})dx + C$$

$$y\,(1+x^2) = \int \frac{\cot x}{(1+x^2)}\cdot(1+x^2)dx + C$$

$$\Rightarrow \quad y\,(1+x^2) = \int \cot x + C$$

$$\Rightarrow \quad y\,(1+x^2) = \log|\sin x| + C.$$ **Ans.**

20. Let $\vec{a} = \hat{i}+4\hat{j}+2\hat{k}$, $\vec{b} = 3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$. Find a vector $\vec{p}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and $\vec{p}.\vec{c} = 18$. **[4]**

Solution :

Given that, $\vec{a} = \hat{i}+4\hat{j}+2\hat{k}$, $\vec{b} = 3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c} = 2\hat{i}-\hat{j}+4\hat{k}$

Since $\vec{p}$ is perpendicular to both $\vec{a}$ and $\vec{b}$ therefore $\vec{p}$ is parallel to $(\vec{a}\times\vec{b})$

So, $\quad \vec{p} = \lambda(\vec{a}\times\vec{b})$

$$= \lambda\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 4 & 2 \\ 3 & -2 & 7 \end{vmatrix}$$

$$= \lambda[\hat{i}(28+4)-\hat{j}(7-6)+\hat{k}(-2-12)]$$

$$= \lambda(32\hat{i}-\hat{j}-14\hat{k})$$

Now, $\quad \vec{p}.\vec{c} = 18$

$$\lambda(32\hat{i}-\hat{j}-14\hat{k}).(2\hat{i}-\hat{j}+4\hat{k}) = 18$$

$$\Rightarrow \quad \lambda\,(32\times 2+1-14\times 4) = 18$$

$$\Rightarrow \quad 9\lambda = 18$$

$$\therefore \quad \lambda = 2$$

$$\therefore \quad \vec{p} = 2(32\hat{i}-\hat{j}-14\hat{k})$$

$$\Rightarrow \quad \vec{p} = 64\hat{i}-2\hat{j}-28\hat{k}.$$ **Ans.**

21. Find the coordinates of the point where the line through the points A = (3, 4, 1) and B = (5, 1, 6) crosses the XY-plane. **[4]**

Solution : Given, A = (3, 4, 1), B = (5, 1, 6)

The equation of the line passing through above points is

$$\frac{x-3}{5-3} = \frac{y-4}{1-4} = \frac{z-1}{6-1}$$

$$\Rightarrow \quad \frac{x-3}{2} = \frac{y-4}{-3} = \frac{z-1}{5}$$

This line crosses the XY-plane

$$\therefore \quad z = 0$$

$$\frac{x-3}{2} = \frac{y-4}{-3} = \frac{0-1}{5}$$

Taking 1^{st} and 3^{rd} terms, we get

$$\frac{x-3}{2} = \frac{-1}{5}$$

$$\Rightarrow \quad 5x-15 = -2$$

$$\Rightarrow \quad 5x = 13$$

$$\Rightarrow \quad x = \frac{13}{5}$$

Taking 2^{nd} and 3^{rd} terms, we get

$$\frac{y-4}{-3} = \frac{-1}{5}$$

$$\Rightarrow \quad 5y-20 = 3$$

$$\Rightarrow \quad 5y = 23$$

$$\Rightarrow \quad y = \frac{23}{5}$$

So, coordinates of the point where that line through the points A and B crosses the XY-plane is $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$ **Ans.**

22. Two cards are drawn simultaneously (without replacement) from a well-shuffled pack of 52 cards. Find the mean and variance of the number of red cards. **[4]**

Solution : Let P (A) = Probability of getting one red card.

Number of red cards = 26

Let X be the random variable which can take values 0, 1, 2 where X is the number of red cards selected

$$P(X=0) = \frac{^{26}C_2}{^{52}C_2} = \frac{26\times 25}{52\times 51} = \frac{25}{102}$$

$$P(X=1) = \frac{^{26}C_1\times {^{26}C_1}}{^{52}C_2} = \frac{26\times 26\times 2}{52\times 51} = \frac{52}{102}$$

$$P(X=2) = \frac{^{26}C_2}{^{52}C_2} = \frac{26\times 25}{52\times 51} = \frac{25}{102}$$

Probability distribution of random variable X is

X	0	1	2
P(X)	$\frac{25}{102}$	$\frac{52}{102}$	$\frac{25}{102}$

$$\text{Mean} = \Sigma X P(X) = \frac{52}{102} + \frac{50}{102} = 1$$

$$\text{Variance of } x = \Sigma X^2 P(X) - (\Sigma X P(X))^2$$

$$\Rightarrow \quad \Sigma X^2 P(X) = \frac{52}{102} + \frac{4 \times 25}{102} = \frac{152}{102}$$

$$\Rightarrow \quad (\Sigma X P(X))^2 = 1^2 = 1$$

$$\therefore \quad \text{Variance of } x = \frac{152}{102} - 1 = \frac{50}{102}$$

$$= \frac{25}{51}$$

Ans.

SECTION — C

23. Using matrices, solve the following system of equations:
$2x + 3y + 3z = 5,\ x - 2y + z = -4,\ 3x - y - 2z = 3.$ **[6]**

Solution : Given, $2x + 3y + 3z = 5,$

$$x - 2y + z = -4,$$

$$3x - y - 2z = 3$$

Here, $A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$

$$|A| = 2(4+1) - 3(-2-3) + 3(-1+6)$$

$$= 10 + 15 + 15 = 40 \neq 0$$

$\therefore A^{-1}$ exists.

Co-factors of matrix A is

$A_{11} = 5,\quad A_{12} = 5,\quad A_{13} = 5$

$A_{21} = 3,\quad A_{22} = -13,\quad A_{23} = 11$

$A_{31} = 9,\quad A_{32} = 1,\quad A_{33} = -7$

$$\text{adj } A = \begin{bmatrix} 5 & 5 & 5 \\ 3 & -13 & 11 \\ 9 & 1 & -7 \end{bmatrix}^T = \begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|}$$

$$= \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore \quad X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 25 - 12 + 27 \\ 25 + 52 + 3 \\ 25 - 44 - 21 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$\therefore \quad x = 1, y = 2$ and $z = -1.$ **Ans.**

24. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. **[6]**

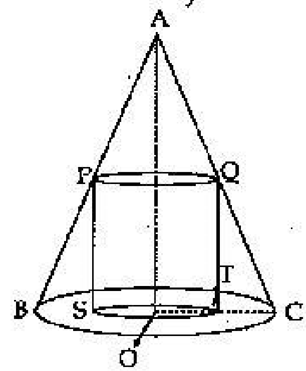
Solution : Let r = radius of cylinder

R = radius of cone

h = height of cylinder

H = height of cone

Curved surface area of cylinder $= 2\pi rh$



In Δ QTC and Δ AOC,

$$\frac{AO}{QT} = \frac{OC}{TC}$$

$$\frac{H}{h} = \frac{R}{R - r}$$

$$\Rightarrow \quad h = \frac{H(R - r)}{R}$$

$$S = 2\pi rh$$

$$\Rightarrow \quad S = 2\pi rH\frac{(R - r)}{R}$$

$$\Rightarrow \quad S = 2\pi H\frac{(Rr - r^2)}{R}$$

Differentiating w.r.t. r, we get

$$\frac{dS}{dr} = \frac{2\pi H}{R}(R - 2r) \quad \text{...(i)}$$

Again differentiating w.r.t. r, we get

$$\frac{d^2S}{dr^2} = \frac{-4\pi H}{R} \quad \text{...(ii)}$$

For maxima and minima,

$$\frac{dS}{dr} = 0$$

From (i),

$$\frac{2\pi H}{R}(R - 2r) = 0$$

$$\Rightarrow \quad R = 2r$$

$$\Rightarrow \quad r = \frac{R}{2}$$

$$\frac{d^2S}{dr^2} = \frac{-4\pi H}{R} < 0$$

Hence, S is maximum when $r = \frac{R}{2}$

Hence Proved.

OR

An open box with a square base is to be made out of a given quantity of cardboard of area c^2 square units. Show that the maximum volume of the box is $\frac{c^3}{6\sqrt{3}}$ cubic units.

Solution : Let $l = x, b = x, h = y$

$\therefore$ Area of open box = Area of cardboard

$$x^2 + 4xy = c^2 \Rightarrow y = \frac{c^2 - x^2}{4x} \quad \text{...(i)}$$

Let V = volume of the box

$$V = x^2 y$$

$$V = x^2\left(\frac{c^2 - x^2}{4x}\right)$$

$$V = \frac{xc^2}{4} - \frac{x^3}{4}$$

Differentiating w.r.t. x, we get

$$\frac{dV}{dx} = \frac{c^2}{4} - \frac{3x^2}{4} \quad \text{...(ii)}$$

Again differentiating w.r.t. x, we get

$$\frac{d^2V}{dx^2} = \frac{-3x}{2}$$

For maxima or minima, $\frac{dV}{dx} = 0$.

$$\frac{c^2}{4} - \frac{3x^2}{4} = 0$$

From (ii),

$$\frac{c^2}{4} = \frac{3x^2}{4}$$

$$\Rightarrow \quad x = \frac{c}{\sqrt{3}}$$

Put the value of x in equation (i), we get

$$y = \frac{c^2 - \frac{c^2}{3}}{4.\frac{c}{\sqrt{3}}}$$

$$\Rightarrow \quad y = \frac{2c^2}{3} \cdot \frac{\sqrt{3}}{4c} = \frac{c}{2\sqrt{3}}$$

Also $\quad \frac{d^2V}{dx^2} = \frac{-3x}{2} < 0$

$\therefore$ Volume is maximum.

$\therefore$ Maximum volume $V = x^2 y$

$$= \frac{c^2}{3} \cdot \frac{c}{2\sqrt{3}} = \frac{c^3}{6\sqrt{3}} \quad \textbf{Ans.}$$

25. Evaluate : $\int \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} dx$. **[6]**

Solution : Let, $I = \int \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx \quad \text{...(i)}$

Let $\quad t = \sin^{-1} x \Rightarrow x = \sin t$

$$dt = \frac{1}{\sqrt{1-x^2}} dx$$

Put the value of dt in equation (i), we get

$$I = \int t \sin t \, dt$$

On integrating by parts method, we get

$$I = t\int \sin t\, dt - \int\left(\frac{d(t)}{dt}\int \sin t\, dt\right)dt$$

$$\Rightarrow \quad I = -t\cos t + \int \cos t\, dt$$

$$\Rightarrow \quad I = -t\cos t + \sin t + c$$

$$\Rightarrow \quad I = -t\sqrt{1-\sin^2 t} + \sin t + c$$

$$= -\sin^{-1} x\sqrt{1-x^2} + x + c$$

$$= x - \sin^{-1} x.\sqrt{1-x^2} + c. \quad \textbf{Ans.}$$

OR

Evaluate : $\int \frac{x^2+1}{(x-1)^2(x+3)} dx$

Solution : Given, $\int \frac{x^2+1}{(x-1)^2(x+3)} dx$

By using partial fractions

$$\frac{x^2+1}{(x-1)^2(x+3)} dx = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+3)}$$

$$x^2 + 1 = A(x-1)(x+3) + B(x+3) + C(x-1)^2$$

Put $x = 1$ in above equation, we get

$$1^2 + 1 = B(1+3)$$

$$\Rightarrow \quad B = \frac{2}{4} = \frac{1}{2}$$

Put $x = -3$ in above equation, we get

$$9 + 1 = C(-3-1)^2$$

$$\Rightarrow \quad C = \frac{10}{16} = \frac{5}{8}$$

Put $x = 0$ in above equation, we get

$$1 = A(-3) + B(3) + C$$

$$\Rightarrow \quad 3A = -1 + \frac{3}{2} + \frac{5}{8} = \frac{-8+12+5}{8} = \frac{9}{8}$$

$$\Rightarrow \quad A = \frac{3}{8}$$

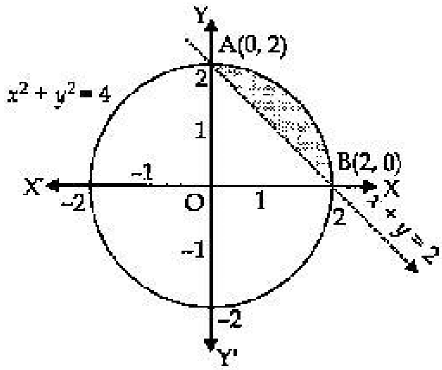
$$\int \frac{x^2+1}{(x-1)^2(x+1)} dx = \frac{3}{8}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{(x-1)^2} + \frac{5}{8}\int \frac{dx}{x+3}$$

$= \frac{3}{8}\log|x-1| - \frac{1}{2(x-1)} + \frac{5}{8}\log|x+3| + c.$ **Ans.**

26. Find the area of the region $\{(x, y) : x^2 + y^2 \le 4, x + y \ge 2\}$. [6]

Solution : We have two equations

$x^2 + y^2 \le 4$ and $x + y \ge 2$



Area under shaded region

$= \text{Area of OBCAO} - \text{Area of } \Delta BOA$

$= \text{Area under circle} - \text{Area under line}$

$$= \int_0^2 (\sqrt{4-x^2})\,dx - \int_0^2 (2-x)\,dx$$

$$= \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2 - \left[2x - \frac{x^2}{2}\right]_0^2$$

$$= \left[\frac{2}{2}\sqrt{4-4} + 2\sin^{-1}\frac{2}{2}\right] - [0 + 2\sin^{-1} 0] - [2\times 2 - 2] + 0$$

$= 2\sin^{-1}(1) - 2.0 - 2$

$= 2\left(\frac{\pi}{2}\right) - 2$

$= (\pi - 2)$ sq. units. **Ans.**

27. If the lines $\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2}$ and $\frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5}$ are perpendicular, find the value of k and hence find the equation of plane containing these lines. [6]

Solution : The equation of the given lines are

$$\frac{x-1}{-3} = \frac{y-2}{-2k} = \frac{z-3}{2} \text{ and } \frac{x-1}{k} = \frac{y-2}{1} = \frac{z-3}{5}$$

These lines are perpendicular, therefore

$$a_1a_2 + b_1b_2 + c_1c_2 = 0$$

$$-3k + (-2k \times 1) + 2 \times 5 = 0$$

$\Rightarrow \qquad -5k = -10$

$\Rightarrow \qquad k = 2$

$\therefore$ The equation of lines becomes

$$\frac{x-1}{-3} = \frac{y-2}{-4} = \frac{z-3}{2}, \quad \frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{5}$$

Equation of plane containing 2 lines is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0,$$

Where $x_1 = 1, \quad y_1 = 2, \quad z_1 = 3$

$l_1 = -3, m_1 = -4, n_1 = 2$

$l_2 = 2, \quad m_2 = 1, \quad n_2 = 5$

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ -3 & -4 & 2 \\ 2 & 1 & 5 \end{vmatrix} = 0$$

$(x-1)(-20-2) - (y-2)(-15-4) + (z-3)(-3+8) = 0$

$\Rightarrow \; -22x + 22 + 19y - 38 + 5z - 15 = 0$

$\Rightarrow \qquad 22x - 19y - 5z + 31 = 0.$ **Ans.**

28. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin 3 times and notes the number of heads. If she gets 1, 2, 3 or 4 she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, what is the probability that she threw 1, 2, 3 or 4 with the die? [6]

Solution : The outcome of an experiment can be represented as

Die is thrown $\begin{cases} \text{Getting 5 or 6} \rightarrow \text{coin is tossed 3 times} \\ \text{Getting 1, 2, 3 or 4} \rightarrow \text{coin is tossed once} \end{cases}$

If she gets 1, 2, 3 or 4, sample space will be (1H), (2H), (3H), (4H), (1T), (2T), (3T), (4T)

If she gets 5 or 6, sample space will be (5 HHH), (5 HTT), (5 HHT), (5 HTH), (5 THT), (5 TTH), (5 THH), (5 TTT), (6 HHH), (6 HTT), (6 HHT), (6 HTH), (6 THT), (6 TTH), (6 THH), (6 TTT)

Let

A = Getting 1, 2, 3 or 4 on die

B = Getting exactly 1 Head

A = (1H), (2H), (3H), (4H), (1T), (2T), (3T), (4T)

B = (1H), (2H), (3H), (4H), (5HTT), (5THT), (5TTH), (6HTT), (6TTH), (6THT)

$A \cap B$ = (1H), (2H), (3H), (4H)

$P(1H) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12} \qquad P(2H) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

$P(1T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12} \qquad P(2T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

$P(3T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$ $\quad$ $P(4T) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$

$P(5TTH) = \frac{1}{6} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{48}$ $\quad$ $P(5THT) = \frac{1}{48}$

$P(5HTT) = \frac{1}{48}$, $P(6THT) = \frac{1}{48}$, $P(6TTH) = \frac{1}{48}$,

$P(6HTT) = \frac{1}{48}$

$$P(A \cap B) = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} = \frac{4}{12} = \frac{1}{3}$$

$P(B) = P(1H) + P(2H) + P(3H) + P(4H) + P(5THT) + P(5HTT) + P(5TTH) + P(6HTT) + P(6THT) + P(6TTH)$

$$= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{48} + \frac{1}{48} + \frac{1}{48} + \frac{1}{48} + \frac{1}{48} + \frac{1}{48}$$

$$= \frac{4}{12} + \frac{6}{48} = \frac{22}{48} = \frac{11}{24}$$

$\therefore$ Required probability is

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{3}}{\frac{11}{24}} = \frac{8}{11}.$$ **Ans.**

29. A dietician wishes to mix two types of foods in such a way that the vitamin contents of the mixture contains at least 8 units of vitamin A and 10 units of vitamin C. Food I contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C while food II contains 1 unit/kg of vitamin A and 2 units/kg of vitamin 1 unit/kg of vitamin C. It costs ₹ 5 per kg to purchase food I and ₹ 7 per kg to purchase Food II. Determine the minimum cost of such a mixture. Formulate the above as a LPP and solve it graphically. **[6]**

Solution :

Vitamins	Food 1	Food 2	Requirement
Vitamin A	2	1	8
Vitamin C	1	2	10
Cost (in ₹)	5	7	

Let the amount of food I = x kg
Let the amount of food II = y kg
If Z denotes the total cost
To minimize the cost we have to minimize Z.
Subject to the constraints,

$$2x + y \geq 8$$
$$x + 2y \geq 10$$
$$x \geq 0$$
$$y \geq 0$$
$$\text{Minimize } Z = 5x + 7y$$

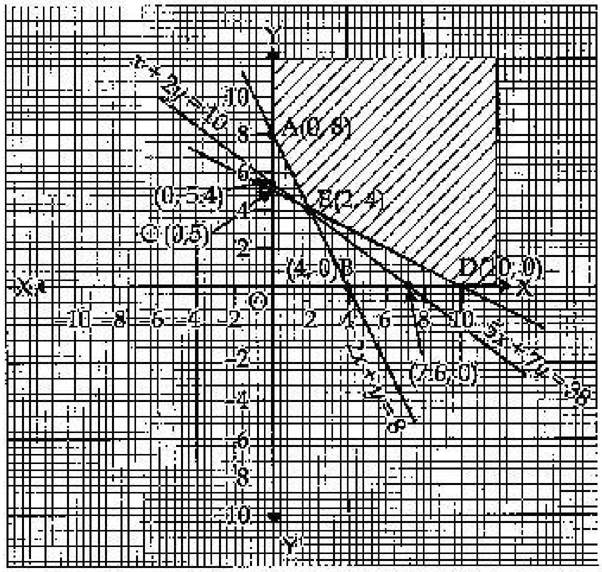
First we draw the lines AB and CD whose equations are

$$2x + y = 8 \quad \text{...(i)}$$

	A	B
x	0	4
y	8	0

$$x + 2y = 10 \quad \text{...(ii)}$$

	C	D
x	0	10
y	5	0



The feasible region is shaded in the figure. The lines are intersecting at the point E (2, 4).

$\therefore$ The vertices of the feasible region are A (0, 8), E (2, 4) and D (10, 0).

Corner points	$Z = 5x + 7y$
At A(0, 8)	$Z = 5(0) + 7(8) = 56$
At E (2, 4)	$Z = 5(2) + 7(4) = 38 \leftarrow$ minimum
At D (10, 0)	$Z = 5(10) + 7(0) = 50$

Since the feasible region is unbounded 38 may or may not be minimum value of total cost for this we draw graph of inequality.

$$5x + 7y < 38$$

x	0	38/5 = 7.6
y	38/7 = 5.4	0

Clearly graph of L has no common point with the feasible region.

$\therefore$ The minimum value of Z is 38 at the point E (2, 4). Hence, the amount of food I is 2 kg and amount of food II is 4 kg should be included in the mixture for minimum cost of ₹ 38. **Ans.**

Time allowed : 3 hours Maximum marks : 100

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — A

10. Write the value of $(\hat{k}\times\hat{j}).\hat{i}+\hat{j}.\hat{k}$ **[1]**

Solution : $(\hat{k}\times\hat{j}).\hat{i}+\hat{j}.\hat{k}$

$$= -(\hat{j}\times\hat{k}).\hat{i}+\hat{j}.\hat{k} \quad (\because \hat{k}\times\hat{j}=-\hat{j}\times\hat{k})$$

$$= -\hat{i}.\hat{i}+\hat{j}.\hat{k} \quad (\because \hat{i}.\hat{i}=1, \hat{j}.\hat{k}=0)$$

$$= -1+0$$

$$= -1.$$ **Ans.**

SECTION — B

19. Prove the following :

$$\cos^{-1}\left(\frac{4}{5}\right)+\cos^{-1}\left(\frac{12}{13}\right)=\cos^{-1}\left(\frac{33}{65}\right)$$ **[4]**

Solution : L.H.S. $= \cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}$

$$= \cos^{-1}\left(\frac{4}{5}\times\frac{12}{13}-\sqrt{1-\left(\frac{4}{5}\right)^2}\sqrt{1-\left(\frac{12}{13}\right)^2}\right)$$

$[\because \cos^{-1}x+\cos^{-1}y=\cos^{-1}(xy-\sqrt{1-x^2}\sqrt{1-y^2})]$

$$= \cos^{-1}\left(\frac{48}{65}-\frac{3}{5}\times\frac{5}{13}\right)=\cos^{-1}\left(\frac{48}{65}-\frac{15}{65}\right)$$

$$= \cos^{-1}\left(\frac{33}{65}\right) = \text{R.H.S.}$$ **Hence Proved.**

20. If $y=(\tan^{-1}x)^2$, show that

$$(x^2+1)^2\frac{d^2y}{dx^2}+2x(x^2+1)\frac{dy}{dx}=2.$$ **[4]**

Solution : Given, $y=(\tan^{-1}x)^2$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = 2\tan^{-1}x.\frac{d}{dx}(\tan^{-1}x)$$

$$= 2\tan^{-1}x.\frac{1}{1+x^2}$$

$$\Rightarrow \quad (x^2+1)\frac{dy}{dx} = 2\tan^{-1}x$$

Again differentiating w.r.t. x, we get

$$(x^2+1).\frac{d^2y}{dx^2}+2x.\frac{dy}{dx}=2.\frac{1}{x^2+1}$$

$$\Rightarrow (x^2+1)^2\frac{d^2y}{dx^2}+2x(x^2+1)\frac{dy}{dx}=2.$$

Hence Proved.

21. Find the particular solution of the differential equation $\frac{dy}{dx}+y\cot x=4x\,\text{cosec}\,x, (x\neq 0)$, given that $y=0$ when $x=\frac{\pi}{2}$ **[4]**

Solution : Given;

$$\frac{dy}{dx}+y\cot x=4x\,\text{cosec}\,x,(x\neq 0) \quad ...(i)$$

On comparing equation (i) with $\frac{dy}{dx}+Py=Q$

Here $P=\cot x$ and $Q=4x\,\text{cosec}\,x$

$$\therefore \quad \text{I.F.} = e^{\int P dx} = e^{\int \cot x dx}$$

$$= e^{\log \sin x} = \sin x$$

$\therefore$ The solution is

$$y(\text{I.F.}) = \int Q.(\text{I.F.})\,dx+C$$

$$y.\sin x = \int 4x\,\text{cosec}\,x.\sin x\,dx+C$$

$$= \int 4x\,dx+C$$

$$\therefore \quad y.\sin x = 2x^2+C \quad ...(ii)$$

Putting $y=0$, when $x=\frac{\pi}{2}$

$$\therefore \quad 0.\sin\frac{\pi}{2} = 2.\left(\frac{\pi}{2}\right)^2+C$$

$$\Rightarrow \quad C = -\frac{\pi^2}{2}$$

Substituting the value of C in equation (ii), we get

$$y.\sin x = 2x^2-\frac{\pi^2}{2}$$

This is the required particular solution of the given differential equation. **Ans.**

22. Find the coordinates of the point where the line through the point (3, –4, –5) and (2, –3, 1) crosses the plane $2x+y+z=7$. **[4]**

Solution : Equation of the line passes through the points (3, – 4, – 5) and (2, – 3, 1) is

$$\frac{x-3}{2-3} = \frac{y+4}{-3+4}=\frac{z+5}{1+5}$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1}=\frac{z+5}{6}=k \text{ (say)} \quad ...(i)$$

$\therefore$ Any point on line is $(-k+3, k-4, 6k-5)$.

Line crosses the plane $2x+y+z=7$ ...(ii)

Point lies on it

$\Rightarrow \quad 2(-k+3)+k-4+6k-5=7$

$\Rightarrow \quad 5k-3=7$

$\Rightarrow \quad k=2$

$\therefore$ The point is $(-2+3, 2-4, 12-5)$

$= (1, -2, 7).$ **Ans.**

SECTION — C

28. Using matrices, solve the following system of equations :

$x+y-z=3;\ 2x+3y+z=10;\ 3x-y-7z=1$ **[6]**

Solution : The given system of equations are

$$x+y-z = 3;$$
$$2x+3y+z = 10;$$
$$3x-y-7z = 1.$$

These equations can be written in matrix form

$$AX = B \qquad ...(i)$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$$

Where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$

Now $\quad |A| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 3 & 1 \\ 3 & -1 & -7 \end{vmatrix}$

$= (-21+1)-(-14-3)-1(-2-9)$

$= -20+17+11 = -20+28 = 8 \neq 0$

$\Rightarrow A^{-1}$ exists.

For adj A,

$A_{11} = -21+1 = -20 \qquad A_{21} = -(-7-1) = 8,$

$A_{12} = -(-14-3) = 17, \qquad A_{22} = -7+3 = -4,$

$A_{13} = -2-9 = -11, \qquad A_{23} = -(-1-3) = 4,$

$A_{31} = 1+3 = 4$

$A_{32} = -(1+2) = -3$

$A_{33} = 3-2 = 1$

$$\text{adj } A = \begin{bmatrix} -20 & 17 & -11 \\ 8 & -4 & 4 \\ 4 & -3 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|}.\text{adj } A = \frac{1}{8}\begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}$$

From (i), $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 10 \\ 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 24 \\ 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad x = 3, y = 1, z = 1.$ **Ans.**

29. Find the length and the foot of the perpendicular from the point P (7, 14, 5) to the plane $2x+4y-z = 2$. Also find the image of point P in the plane. **[6]**

Solution : The given plane is

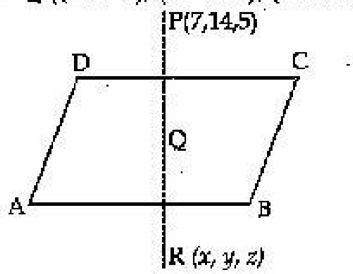
$$2x+4y-z = 2 \qquad ...(i)$$

The d.r.s. of the normal to (i) are $2, 4, -1$.

$\therefore$ Equation of a line perpendicular to (i) passing through P(7, 14, 5) is

$$\frac{x-7}{2} = \frac{y-14}{4} = \frac{z-5}{-1} = k \text{ (say)} \quad ...(ii)$$

$\therefore$ Point is Q $((2k+7), (4k+14), (-k+5))$



Suppose it lies on the plane (i),

$\therefore 2(2k+7)+4(4k+14)-(-k+5) = 2$

$\Rightarrow \quad 21k+65 = 2 \Rightarrow 21k = -63$

$\Rightarrow \quad k = -3.$

$\therefore$ Q $(2\times(-3)+7, 4\times(-3)+14, 3+5)$

$= (1, 2, 8)$

This is the foot of perpendicular of the line (ii) on the plane (i).

$$\Rightarrow \quad PQ = \sqrt{(7-1)^2+(14-2)^2+(5-8)^2}$$

$$= \sqrt{36+144+9} = \sqrt{189}$$

which is the length of the perpendicular from P on (i)

Again let R(x, y, z) be the image of P in the plane (i). Then Q is the mid-point of PR.

∴ The coordinates of Q are given by

$$\left(\frac{x+7}{2}, \frac{y+14}{2}, \frac{z+5}{2}\right)$$

$$\therefore \quad \frac{x+7}{2} = 1, \frac{y+14}{2} = 2, \frac{z+5}{2} = 8$$

$$\Rightarrow \quad x = -5, y = -10, z = 11$$

$\Rightarrow R(-5, -10, 11)$

This is the image of P in the plane (i). **Ans.**

Mathematics 2012 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

> **Note : Except for the following questions, all the remaining questions have been asked in previous sets.**

SECTION–A

9. Write the value of $(\hat{k} \times \hat{i}).\hat{j} + \hat{i}.\hat{k}$ **[1]**

Solution : $(\hat{k} \times \hat{i}).\hat{j} + \hat{i}.\hat{k} = \hat{j}.\hat{j} + 0$

$$\left[\because \hat{k} \times \hat{i} = \hat{j} \text{ and } \hat{i}.\hat{k} = 0\right]$$

$= 1 + 0 \quad [\because \hat{j}.\hat{j} = 1]$

$= 1.$ **Ans.**

10. Find the value of $x + y$ from the following equations :

$$2\begin{pmatrix} 1 & 3 \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$$ **[1]**

Solution : Given,

$$2\begin{pmatrix} 1 & 3 \\ 0 & x \end{pmatrix} + \begin{pmatrix} y & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2 & 6 \\ 0 & 2x \end{pmatrix} + \begin{pmatrix} y & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2+y & 6+0 \\ 1 & 2x+2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 2+y & 6 \\ 1 & 2x+2 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 1 & 8 \end{pmatrix}$$

$$\therefore \quad 2 + y = 5 \Rightarrow y = 3$$

and $\quad 2x + 2 = 8 \Rightarrow 2x = 6 \Rightarrow x = 3$

$$\therefore \quad x + y = 3 + 3 = 6.$$ **Ans.**

SECTION–B

19. If $x = a\left(\cos t + \log \tan \frac{t}{2}\right), y = a \sin t$, find $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ **[4]**

Solution : Given, $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$ and

$y = a \sin t$

Now, $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$

Differentiating both sides w.r. t. t, we get

$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan \frac{t}{2}} \times \sec^2 \frac{t}{2} \times \frac{1}{2}\right)$$

[Applying chain rule of differentiation]

$$= a\left(-\sin t + \frac{\cos \frac{t}{2}}{2\sin \frac{t}{2} \cos^2 \frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{2\sin \frac{t}{2} \cos \frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$

$$\left[\because \sin x = 2\sin \frac{x}{2} \cos \frac{x}{2}\right]$$

$$= a\left(\frac{1 - \sin^2 t}{\sin t}\right)$$

$$\therefore \quad \frac{dx}{dt} = \frac{a\cos^2 t}{\sin t} \quad \text{...(i)}$$

$[\because 1 - \sin^2 t = \cos^2 t]$

Similarly $\quad y = a \sin t$

Differentiating both sides w.r. t. t, we get

$$\frac{dy}{dt} = a \cos t \quad \text{...(ii)}$$

Again, differentiating both sides w.r. t. t, we get

$$\frac{d^2y}{dt^2} = -a \sin t$$

From (i) and (ii), we get

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{\frac{a\cos^2 t}{\sin t}} = \frac{\sin t}{\cos t}$$

$$\therefore \quad \frac{dy}{dx} = \tan t$$

Again differentiating both sides w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(\tan t)$$

$$= \sec^2 t\frac{dt}{dx}$$

$$= \sec^2 t \times \frac{\sin t}{a\cos^2 t}$$

$$= \frac{1}{\cos^2 t} \times \frac{\sin t}{a\cos^2 t} \quad \text{[Using (i)]}$$

$$= \frac{\sin t}{a\cos^4 t}$$ **Ans.**

20. Find the coordinates of the point where the line through the points (3, – 4, – 5) and (2, – 3, 1) crosses the plane 3x + 2y + z + 14 = 0. **[4]**

Solution : The equation of the straight line passing through the points (3, – 4, – 5) and (2, – 3, 1) is

$$\frac{x-3}{2-3} = \frac{y-(-4)}{-3-(-4)} = \frac{z-(-5)}{1-(-5)}$$

$$\left[\because \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}\right]$$

$$\Rightarrow \quad \frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda \text{ (say)}$$

$\Rightarrow \quad x = -\lambda + 3, y = \lambda - 4, z = 6\lambda - 5$

So, the coordinates of a general point on this line are $(-\lambda + 3, \lambda - 4, 6\lambda - 5)$.

The line intersects the given plane $3x + 2y + z + 14 = 0$

$\therefore$ point $(-\lambda + 3, \lambda - 4, 6\lambda - 5)$ lies on the plane

$$3x + 2y + z + 14 = 0$$

$$\Rightarrow 3(-\lambda + 3) + 2(\lambda - 4) + 6\lambda - 5 + 14 = 0$$

$$\Rightarrow \quad -3\lambda + 9 + 2\lambda - 8 + 6\lambda + 9 = 0$$

$$\Rightarrow \quad 5\lambda + 10 = 0$$

$$\Rightarrow \quad \lambda = -2$$

Putting, $\lambda = -2$, we have

$x = -\lambda + 3 = -(-2) + 3 = 5$

$y = \lambda - 4 = -2 - 4 = -6$

$z = 6\lambda - 5 = 6 \times (-2) - 5 = -12 - 5 = -17$

Thus, the point of intersection of the line and the given plane is (5, – 6, – 17). **Ans.**

21. Find the particular solution of the differential equation $x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$, given that when $x = 2$, $y = \pi$. **[4]**

Solution : Given differential equation is :

$$x\frac{dy}{dx} - y + x\sin\left(\frac{y}{x}\right) = 0$$

$$\Rightarrow \quad x\frac{dy}{dx} = y - x\sin\left(\frac{y}{x}\right)$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x\sin\left(\frac{y}{x}\right)}{x} \quad \text{...(i)}$$

Put $y = vx$

Differentiating w.r.t. x, we get

$$\Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substituting the values of y and $\frac{dy}{dx}$ in equation (i), we get

$$v + x\frac{dv}{dx} = \frac{vx - x\sin v}{x}$$

$$\Rightarrow \quad v + x\frac{dv}{dx} = v - \sin v$$

$$\Rightarrow \quad -\frac{dv}{\sin v} = \frac{dx}{x}$$

$$\Rightarrow \quad \text{cosec}\, v\, dv = -\frac{dx}{x}$$

Integrating both sides, we get

$$\log|\text{cosec}\, v - \cot v| = -\log x + \log C = \log\frac{C}{x}$$

$$\Rightarrow \quad \text{cosec}\left(\frac{y}{x}\right) - \cot\left(\frac{y}{x}\right) = \frac{C}{x}$$

$$\Rightarrow \quad \frac{1}{\sin\left(\frac{y}{x}\right)} - \frac{\cos\left(\frac{y}{x}\right)}{\sin\left(\frac{y}{x}\right)} = \frac{C}{x}$$

$$\Rightarrow \quad x\left[1 - \cos\left(\frac{y}{x}\right)\right] = C\sin\left(\frac{y}{x}\right) \quad \text{...(ii)}$$

It is given that when $x = 2, y = \pi$

$$2\left[1 - \cos\left(\frac{\pi}{2}\right)\right] = C\sin\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad 2[1 - 0] = C \times 1$$

$$\Rightarrow \quad C = 2$$

$$\therefore \quad x\left[1 - \cos\left(\frac{y}{x}\right)\right] = 2\sin\left(\frac{y}{x}\right)$$

This is the required particular solution of the given differential equation. **Ans.**

22. Prove the following :

$$\cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)=\sin^{-1}\left(\frac{56}{65}\right)$$ **[4]**

Solution : Taking L.H.S.

$$= \cos^{-1}\left(\frac{12}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$=\sin^{-1}\left(\sqrt{1-\left(\frac{12}{13}\right)^2}\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$\left(\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}\right)$$

$$=\sin^{-1}\left(\sqrt{\frac{25}{169}}\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$=\sin^{-1}\left(\frac{5}{13}\right)+\sin^{-1}\left(\frac{3}{5}\right)$$

$$=\sin^{-1}\left(\frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2}+\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}\right)$$

$$\left(\because \sin^{-1}x+\sin^{-1}y=\sin^{-1}\left(x\sqrt{1-y^2}+y\sqrt{1-x^2}\right)\right)$$

$$=\sin^{-1}\left(\frac{5}{13}\sqrt{\frac{16}{25}}+\frac{3}{5}\sqrt{\frac{144}{169}}\right)$$

$$=\sin^{-1}\left(\frac{5}{13}\cdot\frac{4}{5}+\frac{3}{5}\cdot\frac{12}{13}\right)$$

$$=\sin^{-1}\left(\frac{20}{65}+\frac{36}{65}\right)$$

$$=\sin^{-1}\left(\frac{56}{65}\right)=\text{R.H.S.}$$

Hence Proved.

SECTION–C

28. Find the coordinates of the foot of perpendicular and the length of the perpendicular drawn from the point P (5, 4, 2) to the line $\vec{r}=-\hat{i}+3\hat{j}+\hat{k}+\lambda(2\hat{i}+3\hat{j}-\hat{k})$. Also find the image of P in this line. **[6]**

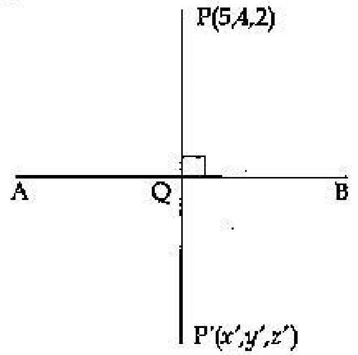
Solution : The given point is (5, 4, 2) and the given line is $\vec{r}=-\hat{i}+3\hat{j}+\hat{k}+\lambda(2\hat{i}+3\hat{j}-\hat{k})$.

This line passes through the point (– 1, 3, 1) and is parallel to the vector $(2\hat{i}+3\hat{j}-\hat{k})$. Cartesian equation of line is

$$\frac{x+1}{2}=\frac{y-3}{3}=\frac{z-1}{-1}=\lambda \text{ (say)}$$

$\therefore x=2\lambda-1, y=3\lambda+3, z=-\lambda+1$

These are the coordinates of any general point on the line.



Let Q be the foot of the perpendicular on the line. Then, for some value of λ, the coordinates of Q are $(2\lambda-1, 3\lambda+3, -\lambda+1)$.

Direction ratios of PQ are $2\lambda-1-5, 3\lambda+3-4, -\lambda+1-2$ *i.e.*, $2\lambda-6, 3\lambda-1, -\lambda-1$.

PQ is perpendicular to given line, we have

$\therefore \quad 2(2\lambda-6)+3(3\lambda-1)-1(-\lambda-1)=0$

$\Rightarrow \quad 4\lambda-12+9\lambda-3+\lambda+1=0$

$\Rightarrow \quad 14\lambda-14=0$

$\Rightarrow \quad \lambda=1$

$\therefore$ Coordinates of Q $=(2\times1-1, 3\times1+3, -1+1)$
$=(1, 6, 0)$

Hence, the coordinates of the foot of perpendicular are (1, 6, 0).

Using distance formula, we have

$$PQ=|\vec{PQ}|=\sqrt{(1-5)^2+(6-4)^2+(0-2)^2}$$
$$=\sqrt{16+4+4}$$
$$=\sqrt{24}$$
$$=2\sqrt{6} \text{ units}$$

Let $P'(x', y', z')$ be the image of point P (5, 4, 2) in the given line.

Then, Q is mid-point of PP′

$\therefore \quad \frac{x'+5}{2}=1\Rightarrow x'+5=2\Rightarrow x'=-3$

$\frac{y'+4}{2}=6\Rightarrow y'+4=12\Rightarrow y'=8$

$\frac{z'+2}{2}=0\Rightarrow z'+2=0\Rightarrow z'=-2$

Hence, the image of P in the given line is (– 3, 8, – 2). **Ans.**

29. Using matrices, solve the following system of equations:

$3x+4y+7z=4; 2x-y+3z=-3; x+2y-3z=8$ **[6]**

Solution : The given system of equation is

$$3x+4y+7z=4;$$
$$2x-y+3z=-3;$$

$$x + 2y - 3z = 8$$

The above system of equations can be represented as

$$AX = B$$

$$\begin{bmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

Here, $A = \begin{bmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{vmatrix}$$

$$= 3(3-6) - 4(-6-3) + 7(4+1)$$
$$= 3 \times (-3) - 4 \times (-9) + 7 \times 5$$
$$= -9 + 36 + 35$$
$$= 62 \neq 0$$

$\therefore A^{-1}$ exists.

So, the given system of equations has a unique solution given by $X = A^{-1}B$

Let A_{ij} be the cofactors of elements a_{ij} in A.

$A_{11} = (3-6) = -3$, $A_{12} = -(-6-3) = 9$,

$A_{13} = (4+1) = 5$

$A_{21} = -(-12-14) = 26$, $A_{22} = (-9-7) = -16$,

$A_{23} = -(6-4) = -2$

$A_{31} = (12+7) = 19$, $A_{32} = -(9-14) = 5$,

$A_{33} = (-3-8) = -11$

$$\text{Adj } A = \begin{bmatrix} -3 & 9 & 5 \\ 26 & -16 & -2 \\ 19 & 5 & -11 \end{bmatrix}^T = \begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \text{Adj } A$$

$$\Rightarrow A^{-1} = \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} -12-78+152 \\ 36+48+40 \\ 20+6-88 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{62}\begin{bmatrix} 62 \\ 124 \\ -62 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence, $x = 1, y = 2, z = -1$ is the required solution.

Ans.

Mathematics 2012 (Delhi) SET I

Time allowed : 3 hours **Maximum marks : 100**

SECTION–A

1. **If a line has direction ratios 2, – 1, – 2, then what are its direction cosines ?** **[1]**

Solution : Given direction ratios are 2, – 1, – 2

i.e., $a = 2, b = -1, c = -2$.

Direction cosines are :

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} = \frac{2}{\sqrt{4+1+4}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$$

$$m = \frac{b}{\sqrt{a^2+b^2+c^2}} = \frac{-1}{\sqrt{4+1+4}} = \frac{-1}{\sqrt{9}} = \frac{-1}{3}$$

$$n = \frac{c}{\sqrt{a^2+b^2+c^2}} = \frac{-2}{\sqrt{4+1+4}} = \frac{-2}{3}$$

$\therefore$ Direction cosines are $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$. **Ans.**

2. **Find 'λ' when the projection of $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$ on $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$ is 4 units.** **[1]**

Solution : Given vectors are, $\vec{a} = \lambda\hat{i} + \hat{j} + 4\hat{k}$, $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$

The projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a}.\vec{b}}{|\vec{b}|}$$

$$\vec{a}.\vec{b} = (\lambda\hat{i} + \hat{j} + 4\hat{k}).(2\hat{i} + 6\hat{j} + 3\hat{k})$$
$$= 2\lambda + 6 + 12$$

$$= 2\lambda + 18$$

$$|\vec{b}| = \sqrt{2^2+6^2+3^2}$$

$$= \sqrt{49} = 7$$

$$\therefore \quad \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = 4$$

$$\Rightarrow \quad \frac{2\lambda+18}{7} = 4$$

$$\Rightarrow \quad 2\lambda + 18 = 28$$

$$\Rightarrow \quad 2\lambda = 10$$

$$\Rightarrow \quad \lambda = 5$$

Ans.

3. Find the sum of the vectors $\vec{a} = \hat{i}-2\hat{j}+\hat{k}$, $\vec{b} = -2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c} = \hat{i}-6\hat{j}-7\hat{k}$. **[1]**

Solution : Given vectors are $\vec{a} = \hat{i}-2\hat{j}+\hat{k}$, $\vec{b} = -2\hat{i}+4\hat{j}+5\hat{k}$ and $\vec{c} = \hat{i}-6\hat{j}-7\hat{k}$.

$$\vec{a}+\vec{b}+\vec{c} = \hat{i}-2\hat{j}+\hat{k}-2\hat{i}+4\hat{j}+5\hat{k}+\hat{i}-6\hat{j}-7\hat{k}$$

$$= 0\hat{i}-4\hat{j}-\hat{k}.$$

Ans.

4. Evaluate : $\int_2^3 \frac{1}{x}dx$. **[1]**

Solution : Given, $\int_2^3 \frac{1}{x}dx$

$$= [\log x]_2^3 \quad \left[\because \int \frac{1}{x}dx = \log x\right]$$

$$= \log 3 - \log 2$$

$$= \log\left(\frac{3}{2}\right)$$

Ans.

5. Evaluate : $\int (1-x)\sqrt{x}\,dx$. **[1]**

Solution : Let, $t = \sqrt{x}$

$$\Rightarrow \quad t^2 = x$$

Differentiating on both sides w.r.t. 'x', we get

$$2tdt = dx$$

$$= \int 2(1-t^2)t^2 dt$$

$$= 2\left[\int t^2 dt - \int t^4 dt\right]$$

$$= 2\left[\frac{t^3}{3} - \frac{t^5}{5}\right] + C$$

Put, $t = \sqrt{x}$

$$= 2\frac{x^{3/2}}{3} - 2\frac{x^{5/2}}{5} + C.$$

Ans.

6. If $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$, write the minor of the element a_{23}. **[1]**

Solution : Let, $A = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$

Minor of the element

$$a_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix}$$

$$= 10 - 3$$

$$= 7.$$

Ans.

7. If $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$, write the value of x. **[1]**

Solution : $\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$

$$\begin{pmatrix} 2-6 & -6+12 \\ 5-14 & -15+28 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$$

$$\begin{pmatrix} -4 & 6 \\ -9 & 13 \end{pmatrix} = \begin{pmatrix} -4 & 6 \\ -9 & x \end{pmatrix}$$

Comparing both sides, we get $x = 13$. **Ans.**

8. Simplify :

$$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$ **[1]**

Solution :

$$\cos\theta\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta\begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$$

Ans.

9. Write the principal value of

$$\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$$ **[1]**

Solution : $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$

$$= \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$$

$[\because \sin^{-1}(-x) = -\sin^{-1} x]$

$$= \frac{\pi}{3} + 2\frac{\pi}{6}$$

$$= \frac{\pi}{3} + \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

Ans.

10. Let * be a 'binary' operation of N given by $a * b$ = LCM (a, b) for all $a, b \in$ N. Find 5 * 7.** **[1]**

SECTION–B

11. If $(\cos x)^y = (\cos y)^x$, find $\frac{dy}{dx}$. **[4]**

Solution : Given, $(\cos x)^y = (\cos y)^x$

Taking log on both sides, we get

$$\log(\cos x)^y = \log(\cos y)^x$$

$$\Rightarrow \quad y \log \cos x = x \log \cos y$$

On differentiating w.r.t. x, we get

$$y.\frac{1}{\cos x}(-\sin x) + \log \cos x.\frac{dy}{dx}$$

$$= x\frac{1}{\cos y}(-\sin y)\frac{dy}{dx} + \log \cos y$$

$$\Rightarrow -y\tan x + \log\cos x\frac{dy}{dx} = -x\tan y\frac{dy}{dx} + \log\cos y$$

$$\Rightarrow (\log\cos x + x\tan y)\frac{dy}{dx} = \log\cos y + y\tan x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log\cos y + y\tan x}{\log\cos x + x\tan y}$$

Ans.

OR

If $\sin y = x\sin(a+y)$, prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

Solution : Given, $\sin y = x\sin(a+y)$

$$\Rightarrow \quad x = \frac{\sin y}{\sin(a+y)}$$

Differentiating both sides w.r.t. y

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}$$

$$= \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$[\because \sin(A-B) = \sin A\cos B - \cos A\sin B]$

$$= \frac{\sin a}{\sin^2(a+y)}$$

Taking reciprocal

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Hence Proved.

12. How many times must a man toss a fair coin, so that the probability of having at least one head is more than 80%? **[4]**

Solution : Let us consider,

x = Number of times a man should toss a coin.

P(H) = Probability of getting atleast one head.

If $x = 1$, sample space will be H, T

$$\therefore \quad P(H) = \frac{1}{2} = 50\%$$

If $x = 2$, sample space will be HH, HT, TH, TT

$$\text{Therefore,} \quad P(H) = \frac{3}{4} = 75\%$$

If $x = 3$, sample space will be HHH, HHT, HTH, HTT, TTT, TTH, THH, THT.

$$\therefore \quad P(H) = \frac{7}{8} = 87.5\% > 80\%$$

Hence, a coin should be tossed 3 times in order to have the probability of getting atleast one head is more than 80%. **Ans.**

13. Find the vector and cartesian equations of the line passing through the point (1, 2, – 4) and perpendicular to the two lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$. **[4]**

Solution : Let, the direction ratios of required line be p, q, r and given that, line is perpendicular to two given lines,

Therefore, we get

$$3p - 16q + 7r = 0$$

$$3p + 8q - 5r = 0$$

On solving, we get

$$\frac{p}{80-56} = \frac{q}{-(-15-21)} = \frac{r}{24+48}$$

$$\Rightarrow \quad \frac{p}{24} = \frac{q}{36} = \frac{r}{72}$$

$$\text{or} \quad \frac{p}{2} = \frac{q}{3} = \frac{r}{6}$$

The required line passing through (1, 2, – 4) has direction ratios proportional to 2, 3, 6.

So cartesian equation of line is

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$$

Equation of line passing through (1, 2, – 4) has

**Answer is not given due to the change in present syllabus

direction ratios, proportional to 2, 3, 6.

i.e. in vector form, this line passes through point having position vector,

$$\vec{a} = \hat{i}+2\hat{j}-4\hat{k}$$

This is parallel to vector

$$\vec{b} = 2\hat{i}+3\hat{j}+6\hat{k}$$

Therefore, vector form of line will be

$$\vec{r} = \vec{a}+\lambda.\vec{b}$$

$$\Rightarrow \quad = (\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k}).$$

Ans.

14. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $|\vec{a}|=5, |\vec{b}|$ = 12 and $|\vec{c}|=13$ and $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$. **[4]**

Solution : Given that, $|\vec{a}|=5, |\vec{b}|=12, |\vec{c}|=13$

and $\vec{a}+\vec{b}+\vec{c}=0$

We know that,

$$|\vec{a}+\vec{b}+\vec{c}|^2$$

$$= |\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$$

$$\Rightarrow 0 = 25+144+169+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$$

$$\Rightarrow (\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}) = \frac{-338}{2}$$

$$= -169.$$

Ans.

15. Solve the following differential equation :

$$2x^2\frac{dy}{dx}-2xy+y^2=0.$$ **[4]**

Solution : The given differential equation is :

$$2x^2\frac{dy}{dx}-2xy+y^2=0$$

$$\Rightarrow \quad 2x^2\frac{dy}{dx} = 2xy-y^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2xy-y^2}{2x^2} \quad \text{...(i)}$$

Put, $y = vx$

Differentiate w.r.t. 'x' on both sides, we get

$$\frac{dy}{dx} = v+\frac{xdv}{dx}$$

From (i)

$$\Rightarrow \quad v+\frac{xdv}{dx} = \frac{2x^2v-v^2x^2}{2x^2}$$

$$\Rightarrow \quad v+\frac{xdv}{dx} = \frac{2v-v^2}{2}$$

$$\Rightarrow \quad \frac{xdv}{dx} = \frac{2v-v^2}{2}-v$$

$$\Rightarrow \quad \frac{xdv}{dx} = -\frac{v^2}{2}$$

$$\frac{-1}{v^2}dv = \frac{1}{2x}dx$$

Integrating both sides, we get

$$-\int\frac{1}{v^2}dv = \int\frac{1}{2x}dx$$

$$\frac{1}{v} = \frac{1}{2}\log|x|+C$$

$$\frac{1}{\frac{y}{x}} = \frac{1}{2}\log|x|+C$$

$$\frac{x}{y} = \frac{1}{2}\log x+C$$

$$2x = y\log x+2yC.$$

Ans.

16. Find the particular solution of the following differential equation:

$\frac{dy}{dx}=1+x^2+y^2+x^2y^2$, given that $y = 1$ when $x = 0$. **[4]**

Solution : Given, $\frac{dy}{dx} = 1+x^2+y^2+x^2y^2$, $y = 1, x = 0$.

$$\frac{dy}{dx} = (1+x^2)(1+y^2)$$

$$\Rightarrow \quad \frac{dy}{1+y^2} = (1+x^2)\,dx$$

Integrating both sides

$$\int\frac{dy}{1+y^2} = \int(1+x^2)dx$$

$$\Rightarrow \quad \tan^{-1}y = x+\frac{x^3}{3}+C$$

and given, $y = 1, x = 0$

$$\tan^{-1}(1) = 0+C$$

$$\Rightarrow \quad \tan^{-1}(1) = C$$

$$\Rightarrow \quad C = \frac{\pi}{4}$$

$$\therefore \quad \tan^{-1}y = x+\frac{x^3}{3}+\frac{\pi}{4}$$

Ans.

17. Evaluate : $\int\sin x\sin 2x\sin 3x\,dx$. **[4]**

Solution : $\int\sin x\sin 2x\sin 3x\,dx$

Multiply and divide by 2,

$$= \frac{1}{2}\int(2\sin x\sin 2x)\sin 3x\,dx$$

$$= \frac{1}{2}\int(\cos x - \cos 3x)\sin 3x\,dx$$

$$[\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)]$$

$$= \frac{1}{2}\int(\sin 3x\cos x - \cos 3x\sin 3x)\,dx$$

$$= \frac{1}{2}\int(\sin 3x\cos x - \cos 3x\sin 3x)\,dx$$

$$= \frac{1}{4}\int(\sin 4x + \sin 2x - \sin 6x)\,dx$$

$$[\ 2\sin A\cos B = \sin(A+B) + \sin(A-B)]$$

$$= \frac{1}{4}\left[-\frac{\cos 4x}{4} - \frac{\cos 2x}{2} + \frac{\cos 6x}{6}\right] + C.$$ **Ans.**

OR

Evaluate : $\int \frac{2}{(1-x)(1+x^2)}dx.$

Solution : By method of partial fractions :

$$\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{x^2+1}$$

Now, $2 = A(x^2+1) + (Bx+C)(1-x)$

Putting $x = 1$, we get

$$2 = 2A + (B+C)\,0$$
$$\Rightarrow \quad 2 = 2A$$
$$\Rightarrow \quad A = 1$$

Putting $x = 0$, we get

$$2 = A + C,$$
$$\Rightarrow \quad 2 = 1 + C$$
$$\Rightarrow \quad C = 1$$

Putting $x = -1$, we get

$$2 = 2A + 2(-B+C)$$
$$\Rightarrow \quad 2 = 2 + 2(-B+1)$$
$$\Rightarrow \quad -B + 1 = 0$$
$$\Rightarrow \quad B = 1.$$

$$\int \frac{2}{(1-x)(1+x^2)}dx = \int \frac{A}{1-x}dx + \int \frac{Bx+C}{x^2+1}dx$$

$$= \int \frac{1}{1-x}dx + \int \frac{x+1}{x^2+1}dx$$

$$= \int \frac{1}{1-x}dx + \frac{1}{2}\int \frac{2x}{1+x^2}dx + \int \frac{1}{1+x^2}dx$$

$$= -\log|x-1| + \frac{1}{2}\log(1+x^2) + \tan^{-1}x + C.$$ **Ans.**

18. Find the point on the curve $y = x^3 - 11x + 5$ at which the equation of tangent is $y = x - 11$. [4]

Solution : Given curve is $y = x^3 - 11x + 5$ ...(i)

Slope of tangent to curve

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 - 11$$

and tangent is $y = x - 11$ ...(ii)

$$\Rightarrow \quad x - y - 11 = 0$$

Slope of tangent,

$$\frac{dy}{dx} = 1$$

Equating slopes,

$$3x^2 - 11 = 1$$
$$\Rightarrow \quad 3x^2 = 12$$
$$\Rightarrow \quad x^2 = 4$$
$$\Rightarrow \quad x = \pm 2.$$

put, $x = 2$, in (i),

$$y = (2)^3 - 11 \times 2 + 5$$
$$= 8 - 22 + 5$$
$$= -9$$

put $x = -2$ in (i),

$$y = (-2)^3 - 11(-2) + 5$$
$$\Rightarrow \quad y = -8 + 22 + 5$$
$$\Rightarrow \quad y = 19$$

The points on the curve are, $(2, -9)$, $(-2, 19)$

Now, put these values in (ii)

$$-9 = 2 - 11$$

So, $(2, -9)$ is satisfying the tangent equation.

But $(-2, 19)$ does not satisfy tangent equation.

Hence $(2, -9)$ is the required point on curve.

Ans.

OR

Using differentials, find the approximate value of $\sqrt{49.5}$.

Solution : Let $y = \sqrt{x} = \sqrt{49}$ ...(i)

$$y + \Delta y = \sqrt{x + \Delta x} = \sqrt{49.5} \quad \text{...(ii)}$$

(ii) – (i) gives

$$\Delta y = \sqrt{49.5} - \sqrt{49}$$
$$= \sqrt{49.5} - 7$$

$$\therefore \quad \sqrt{49.5} = \Delta y + 7 \quad \text{...(iii)}$$

$$\Delta y \approx dy = \frac{dy}{dx}.\Delta x = \frac{1}{2\sqrt{x}} \times 0.5$$

$$\left[\because y = \sqrt{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x}}\right]$$

$$= \frac{1}{2 \times 7} \times 0.5 = 0.036$$

Put value in eq. (iii)

$$\sqrt{49.5} = 0.036 + 7$$
$$= 7.036.$$ **Ans.**

19. If $y = (\tan^{-1}x)^2$, show that

$$(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2.$$ **[4]**

Solution : Given, $y = (\tan^{-1} x)^2$

Differentiating w.r.t. 'x' on both sides

$$\frac{dy}{dx} = 2\tan^{-1} x.\frac{1}{x^2+1}$$

$$\Rightarrow \quad (x^2+1)\frac{dy}{dx} = 2\tan^{-1} x$$

$$\Rightarrow \quad \frac{dy}{dx}x^2 + \frac{dy}{dx} = 2\tan^{-1} x$$

Again differentiating w.r.t. 'x' on both sides

$$\frac{dy}{dx}\times 2x + x^2\frac{d^2y}{dx^2} + \frac{d^2y}{dx^2} = \frac{2}{1+x^2}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{2}{1+x^2}$$

$$\Rightarrow (1+x^2)^2\frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} = 2.$$ **Hence Proved.**

20. Using properties of determinants, prove that

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}.$$ **[4]**

Solution : Given,

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

$$\text{L.H.S.} = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= 2\begin{vmatrix} a+b+c & p+q+r & x+y+z \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$

$$= 2\begin{vmatrix} a & p & x \\ -b & -q & -y \\ -c & -r & -z \end{vmatrix}$$

Taking –1 common from R_2 and R_3

$$= 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \text{R.H.S.}$$ **Hence Proved.**

21. Prove that : $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right) = \frac{\pi}{4} - \frac{x}{2}, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **[4]**

Solution :

$$\text{L.H.S.} = \tan^{-1}\left[\frac{\cos x}{1+\sin x}\right]$$

$$= \tan^{-1}\left[\frac{\cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right) + \sin^2\left(\frac{x}{2}\right) + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right]$$

$$\left[\begin{array}{l} \because \quad \cos 2\theta = \cos^2\theta - \sin^2\theta \\ \quad\quad \cos^2\theta + \sin^2\theta = 1 \\ \text{and } \sin 2\theta = 2\sin\theta\cos\theta \end{array}\right]$$

$$= \tan^{-1}\left(\frac{\left\{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right\}\left\{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right\}}{\left\{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\right\}^2}\right)$$

$$[\because a^2 - b^2 = (a-b)(a+b)]$$

$$= \tan^{-1}\left[\frac{\left\{\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\right\}}{\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)}\right]$$

Dividing Numerator & Denominator with $\cos\left(\frac{x}{2}\right)$.

$$= \tan^{-1}\left[\frac{1 - \tan\left(\frac{x}{2}\right)}{1 + \tan\left(\frac{x}{2}\right)}\right]$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} - \tan\left(\frac{x}{2}\right)}{\tan\frac{\pi}{4}\tan\left(\frac{x}{2}\right) + 1}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} - \frac{x}{2}\right)\right)$$

$$= \frac{\pi}{4} - \frac{x}{2} = \text{R.H.S.}$$ **Hence Proved.**

OR

Prove that $\sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \cos^{-1}\left(\frac{36}{85}\right)$.

Solution :

$$\text{L.H.S.} = \sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

and we know that

$$\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + (\sqrt{1-x^2})y\right)$$

$= \sin^{-1}\left(\frac{8}{17}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$= \sin^{-1}\left\{\frac{8}{17}\sqrt{1-\left(\frac{3}{5}\right)^2} + \frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2}\right\}$

$= \sin^{-1}\left\{\frac{8}{17}\sqrt{\frac{16}{25}} + \frac{3}{5}\sqrt{\frac{225}{289}}\right\}$

$= \sin^{-1}\left\{\frac{8}{17}\times\frac{4}{5} + \frac{3}{5}\times\frac{15}{17}\right\}$

$= \sin^{-1}\left(\frac{77}{85}\right) = \cos^{-1}\left(\sqrt{1-\left(\frac{77}{85}\right)^2}\right)$

[$\because \sin^{-1}x = \cos^{-1}\sqrt{1-x^2}$]

$= \cos^{-1}\left(\sqrt{\frac{1296}{(85)^2}}\right)$

$= \cos^{-1}\left(\frac{36}{85}\right) = \text{R.H.S.}$ **Hence Proved.**

22. Let A = R – {3} and B = R – {1}. Consider the function f : A → B defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Show that f is one-one and onto and hence find f^{-1}. [4]

Solution : Let $x, y \in R$ such that

$$f(x) = f(y)$$

$$\Rightarrow \frac{x-2}{x-3} = \frac{y-2}{y-3}$$

$$\Rightarrow xy - 3x - 2y + 6 = xy - 2x - 3y + 6$$

$$\Rightarrow x = y$$

∴ Function is one-one.

Let, $y = f(x)$

$$y = \frac{x-2}{x-3}$$

$$\Rightarrow x - 2 = xy - 3y$$

$$\Rightarrow x = \frac{3y-2}{y-1}$$

Now, consider,

$$f\left(\frac{3y-2}{y-1}\right) = \frac{\frac{3y-2}{y-1}-2}{\frac{3y-2}{y-1}-3}$$

$$= \frac{3y-2-2y+2}{3y-2-3y+3}$$

$$= y$$

Since $y \neq 1$ and $\frac{3y-2}{y-1} \neq 3$

$\therefore x \in A$

∴ for value $y \in B$, there exists $x = \frac{3y-2}{y-1}$

Such that $f(x) = y$

$\Rightarrow f : A \to B$ is onto. **Hence Proved.**

Now, $f(x) = y \Rightarrow x = f^{-1}(y)$ ...(i)

$$\frac{x-2}{x-3} = y$$

$$\Rightarrow x - 2 = xy - 3y$$

$$\Rightarrow x = \frac{3y-2}{y-1}$$

From (i),

$$f^{-1}(y) = \frac{3y-2}{y-1}$$

Thus, $f : A \to B$ is defined as for all $x \in A$

$$f^{-1}(x) = \frac{3x-2}{x-1}$$ **Ans.**

SECTION–C

23. Find the equation of the plane determined by the points A (3, – 1, 2), B (5, 2, 4) and C (–1, – 1, 6) and hence find the distance between the plane and the point P (6, 5, 9). [6]

Solution : Any plane passes through A is,

$$a(x-3) + b(y+1) + c(z-2) = 0 \quad \text{...(i)}$$

Plane (i) passes through B and C

$$a(5-3) + b(2+1) + c(4-2) = 0$$

$$\Rightarrow 2a + 3b + 2c = 0$$

$$a(-1-3) + b(-1+1) + c(6-2) = 0$$

$$\Rightarrow -4a + 4c = 0$$

On solving equations, we get

$$\frac{a}{12} = \frac{b}{-16} = \frac{c}{12}$$

Let, $\frac{a}{3} = \frac{b}{-4} = \frac{c}{3} = k$ (say)

$$a = 3k, b = -4k, c = 3k$$

Putting the value of a, b and c in equation (i), we get

$$3k(x-3) - 4k(y+1) + 3k(z-2) = 0$$

$$\Rightarrow 3(x-3) - 4(y+1) + 3(z-2) = 0$$

$$\Rightarrow 3x - 4y + 3z = 19$$

Thus the required equation of the plane is $3x - 4y + 3z = 19$

The distance of point (6, 5, 9) from the equation of plane $3x - 4y + 3z - 19 = 0$ is,

$$\frac{|3\times6-4\times5+3\times9-19|}{\sqrt{3^2+(-4)^2+3^2}} = \frac{|18-20+27-19|}{\sqrt{9+16+9}}$$

$$= \frac{6}{\sqrt{34}}. \quad \textbf{Ans.}$$

24. Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in hostel). Previous year results report that 30% of all students who reside in hostel attain 'A' grade and 20% of dayscholars attain 'A' grade in their annual examination. At the end of the year, one student is chosen at random from the college and he has an 'A' grade, what is the probability that the student is a hostler? [6]

Solution : Let the events be defined as

E_1 = Students reside in hostel

E_2 = Selected student is a day scholar

A = Getting "A" grade

and $P(E_1) = 0.60$

$P(E_2) = 0.40$

$P(A/E_1) = 0.30$

$P(A/E_2) = 0.20$

We know that by Bayes' theorem

$$P(E_1/A) = \frac{P(E_1).P(A/E_1)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)}$$

$$= \frac{0.60 \times 0.30}{0.60 \times 0.30 + 0.40 \times 0.20}$$

$$= \frac{0.18}{0.26} = \frac{9}{13} = 0.69 \quad \textbf{Ans.}$$

25. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of ₹ 17.50 per package on nuts and ₹ 7 per package of bolts. How many packages of each should be produced each day so as to maximize his profits if he operate his machines for at the most 12 hours a day? Form the above as a linear programming problem and solve it graphically. [6]

Solution :

Machine	Nuts (x)	Bolts (y)	Maximum hrs.
Machine A	1	3	12
Machine B	3	1	12
Cost (in ₹)	17.50	7	

Let Packages of Nuts = x

Packages of Bolt = y

If Z denotes the total cost.

To maximise the cost, we have to maximize Z.

Maximize $Z = 17.50\,x + 7y$

Subject to the constraints

$$x + 3y \le 12$$

$$3x + y \le 12$$

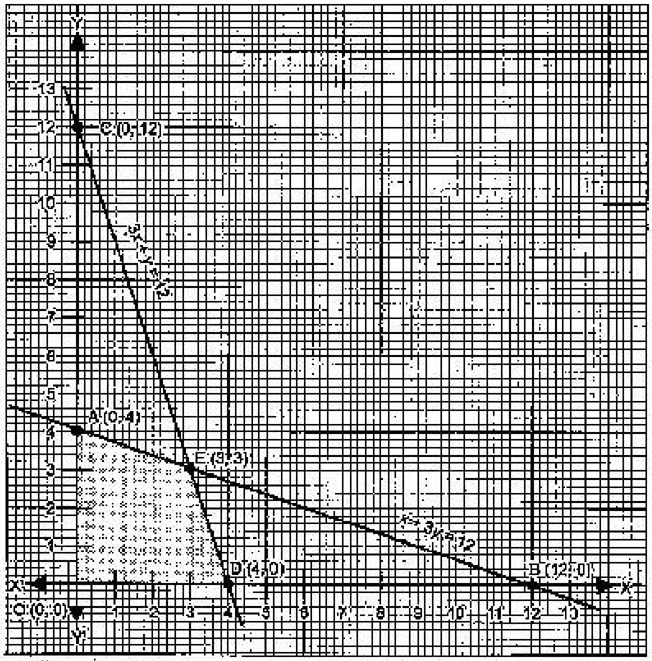
$$x \ge 0, y \ge 0$$

First we draw the lines AB and CD whose equations are

$x + 3y = 12$ ...(i) $\qquad$ $3x + y = 12$ ...(ii)

	A	B
x	0	12
y	4	0

	C	D
x	0	4
y	12	0



The feasible region OAEDO is shaded in the figure.

The lines are intersecting the point E(3, 3).

∴ The vertices of the feasible region are O(0, 0), A (0, 4), E (3, 3) and D(4, 0).

Points	$Z = 17.50x + 7y$
At O(0, 0)	$Z = 17.50(0) + 7(0) = 0$
At A(0, 4)	$Z = 17.50(0) + 7(4) = 28$
At E(3, 3)	$Z = 17.50(3) + 7(3) = 73.50$
At D(4, 0)	$Z = 17.50(4) + 7(0) = 70$

∴ The maximum value of Z is 73.50 at the point E(3, 3).

Hence the 3 packages of nuts and 3 packages of bolt should be produced each day to get the maximum profit of ₹ 73.50. **Ans.**

26. Prove that : $\int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x})\,dx = \sqrt{2}.\frac{\pi}{2}$ **[6]**

Solution : Taking L.H.S.

$$= \int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x})\,dx$$

$$= \int_0^{\pi/4} \left(\sqrt{\frac{\sin x}{\cos x}} + \sqrt{\frac{\cos x}{\sin x}} \right) dx$$

$$= \int_0^{\pi/4} \left(\frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} \right) dx$$

$$= \sqrt{2} \int_0^{\pi/4} \left(\frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}} \right) dx$$

$$= \sqrt{2} \int_0^{\pi/4} \left(\frac{\sin x + \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} \right) dx$$

Let, $t = \sin x - \cos x \Rightarrow dt = (\sin x + \cos x)\, dx$

At $x = 0$, $t = \sin 0 - \cos 0 = -1$

At $x = \frac{\pi}{4}$, $t = \sin \frac{\pi}{4} - \cos \frac{\pi}{4} = 0$

$$= \sqrt{2} \int_{-1}^{0} \frac{dt}{\sqrt{1 - t^2}}$$

$$= \sqrt{2} \left| \sin^{-1} t \right|_{-1}^{0}$$

$$= \sqrt{2}[\sin^{-1}(0) - \sin^{-1}(-1)]$$

$$= \sqrt{2} \sin^{-1}(1)$$

$$= \sqrt{2} . \frac{\pi}{2} = \text{R.H.S.}$$ **Hence Proved.**

OR

Evaluate : $\int_1^3 (2x^2 + 5x)\,dx$ **as a limit of a sum.**

Solution : Given, $\int_1^3 (2x^2 + 5x)\,dx$

We know that

$$\int_a^b f(x)\,dx = \lim_{h \to 0} h[f(a) + f(a+h) + f(a+2h) + \ldots + f\{a + (n-1)h\}],$$

where $h = \frac{b-a}{n}$

Here, $f(x) = 2x^2 + 5x$

$$h = \frac{3-1}{n} = \frac{2}{n}$$

$$\int_1^3 (2x^2 + 5x)\,dx$$

$$= \lim_{h \to 0} h[f(1) + f(1+h) + f(1+2h) + \ldots + f\{1 + (n-1)h\}]$$

$$= \lim_{h \to 0} h[\{2(1)^2 + 5(1)\} + \{2(1+h)^2 + 5(1+h)\} + \{2(1+2h)^2 + 5(1+2h)\} + \ldots + \{2(1+(n-1)h)^2 + 5(1+(n-1)h)\}]$$

$$= \lim_{h \to 0} h[2\{1^2 + (1+h)^2 + (1+2h)^2 + \ldots + (1+(n-1)h)^2\} + 5\{1 + (1+h) + (1+2h) + \ldots + (1+(n-1)h)\}]$$

$$= \lim_{h \to 0} h[2\{n + 2h(1+2+3+\ldots+(n-1)) + h^2(1^2 + 2^2 + \ldots + (n-1)^2)\} + 5\{n + h(1 + 2 + \ldots + (n-1))\}]$$

$$= \lim_{h \to 0} h\left[2\left\{n + 2h.\frac{n(n-1)}{2} + h^2.\frac{n(n-1)(2n-1)}{6}\right\} + \left\{5n + 5h\frac{(n(n-1))}{2}\right\}\right]$$

$$= \lim_{h \to 0} h\left[\left\{2n + 2h.n(n-1) + h^2\frac{n(n-1)(2n-1)}{3}\right\} + \left\{5n + 5h\frac{(n(n-1))}{2}\right\}\right]$$

Put $h = \frac{2}{n}$

$$= \lim_{n \to \infty} \frac{2}{n}\left[\left\{2n + 2.\frac{2}{n}n(n-1) + \left(\frac{2}{n}\right)^2 . \frac{n(n-1)(2n-1)}{3}\right\} + \left\{5n + 5.\frac{2}{n}.\frac{n(n-1)}{2}\right\}\right]$$

$$= \lim_{n \to \infty} \frac{2}{n}\left[\left\{2n + 4(n-1) + \frac{4(n-1)(2n-1)}{3n}\right\} + \left\{5n + 5(n-1)\right\}\right]$$

$$= \lim_{n \to \infty} \frac{2}{n}\left[\left\{6n - 4 + \frac{4(2n^2 - 3n + 1)}{3n}\right\} + (10n - 5)\right]$$

$$= \lim_{n \to \infty} \frac{2}{n}\left[\left\{\frac{18n^2 - 12n + 8n^2 - 12n + 4}{3n}\right\} + (10n - 5)\right]$$

$$= \lim_{n \to \infty} \frac{2}{n}\left[\left(\frac{26n^2 - 24n + 4}{3n}\right) + (10n - 5)\right]$$

$$= \lim_{n \to \infty} \frac{2}{n}\left[\frac{26n^2 - 24n + 4 + 30n^2 - 15n}{3n}\right]$$

$$= \lim_{n \to \infty} \frac{2}{n}\left[\frac{56n^2 - 39n + 4}{3n}\right]$$

$$= \lim_{n\to\infty} \frac{2}{3}\left[\frac{56n^2 - 39n + 4}{n^2}\right]$$

$$= \lim_{n\to\infty} \frac{2}{3}\left[56 - \frac{39}{n} + \frac{4}{n^2}\right]$$

$$= \frac{2}{3}(56)$$

$$= \frac{112}{3}.$$ **Ans.**

27. Using the method of integration, find the area of the region bounded by the lines $3x - 2y + 1 = 0$, $2x + 3y - 21 = 0$ and $x - 5y + 9 = 0$. **[6]**

Solution : Given equations are,

$$3x - 2y + 1 = 0$$
$$2x + 3y - 21 = 0$$
$$x - 5y + 9 = 0$$

Taking, $3x - 2y + 1 = 0$

$$\Rightarrow \quad y = \frac{3x+1}{2} \quad \text{...(i)}$$

x	1	3	0
y	2	5	0.5

$$2x + 3y - 21 = 0$$

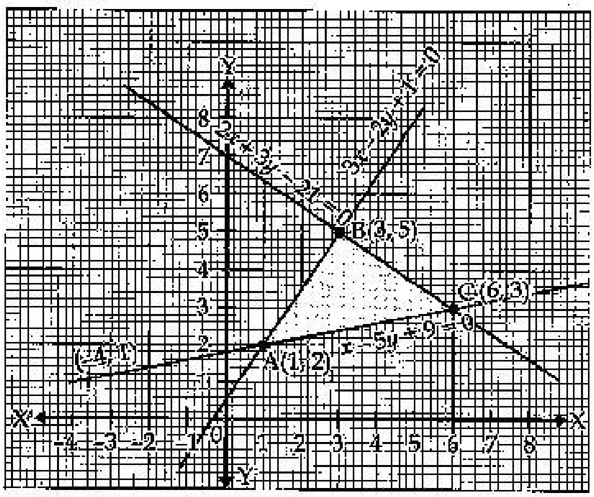
$$\Rightarrow \quad y = \frac{21-2x}{3} \quad \text{...(ii)}$$

x	3	6	0
y	5	3	7

$$x - 5y + 9 = 0$$

$$\Rightarrow \quad y = \frac{x+9}{5} \quad \text{...(iii)}$$

x	1	6	–4
y	2	3	1



The required area of shaded bounded region ABCA

= Area under line AB + Area under line BC – Area under line AC

$$= \int_1^3 \left(\frac{3x+1}{2}\right)dx + \int_3^6 \left(\frac{21-2x}{3}\right)dx - \int_1^6 \left(\frac{x+9}{5}\right)dx$$

$$= \frac{1}{2}\left[\frac{3x^2}{2} + x\right]_1^3 + \frac{1}{3}\left[21x - \frac{2x^2}{2}\right]_3^6 - \frac{1}{5}\left[\frac{x^2}{2} + 9x\right]_1^6$$

$$= \frac{1}{2}\left[\frac{3}{2}(9-1) + 3 - 1\right] + \frac{1}{3}[21(6-3) - (6^2 - 3^2)] - \frac{1}{5}\left[\frac{1}{2}(6^2 - 1^2) + 9(6-1)\right]$$

$$= \frac{1}{2}[[12 + 3 - 1] + \frac{1}{3}[21(3) - (36 - 9)] - \frac{1}{5}\left[\frac{1}{2}(35) + 9(5)\right]$$

$$= \frac{1}{2}(14) + \frac{1}{3}(63 - 27) - \frac{1}{5}\left[\frac{35+90}{2}\right]$$

$$= 7 + \frac{36}{3} - \frac{1}{5}\left(\frac{125}{2}\right)$$

$$= 19 - \frac{25}{2} = \frac{13}{2} \text{ sq. units.}$$ **Ans.**

28. Show that the height of a closed right circular cylinder of given surface and maximum volume, is equal to the diameter of its base. **[6]**

Solution : Let the surface area of cylinder is 'S' and volume is 'V'.

$$S = 2\pi rh + 2\pi r^2$$

$$\Rightarrow \quad h = \frac{S - 2\pi r^2}{2\pi r} \quad \text{...(i)}$$

and $\quad V = \pi r^2 h$

$$\Rightarrow \quad V = \pi r^2 \left[\frac{S - 2\pi r^2}{2\pi r}\right]$$

$$\Rightarrow \quad V = r\left[\frac{S - 2\pi r^2}{2}\right]$$

$$\Rightarrow \quad V = \left[\frac{Sr - 2\pi r^3}{2}\right]$$

Differentiating both sides w.r.t. r, we get

$$\frac{dV}{dr} = \left[\frac{S - 6\pi r^2}{2}\right]$$

For maximum or minimum,

$$\frac{dV}{dr} = 0$$

$$\Rightarrow \quad \frac{S - 6\pi r^2}{2} = 0$$

$$\Rightarrow \quad S = 6\pi r^2$$

Substitute the value of S in equation (i),

$$h = \frac{6\pi r^2 - 2\pi r^2}{2\pi r}$$

$$\Rightarrow \quad h = \frac{4\pi r^2}{2\pi r}$$

$$\Rightarrow \quad h = 2r$$

Again differentiating,

$$\frac{d^2V}{dr^2} = -6\pi r < 0$$

$\therefore$ Volume is maximum when $h = 2r$.

Hence Proved.

29. Using matrices, solve the following system of linear equations :

$$x - y + 2z = 7$$
$$3x + 4y - 5z = -5$$
$$2x - y + 3z = 12 \qquad [6]$$

Solution : Given equations are

$$x - y + 2z = 7$$
$$3x + 4y - 5z = -5$$
$$2x - y + 3z = 12$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$AX = B \qquad ...(i)$$

$$\Rightarrow \quad X = A^{-1}B$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}$$

$$|A| = 1(12-5) + 1(9+10) + 2(-3-8)$$
$$= 7 + 19 - 22$$
$$= 4 \neq 0$$

$\therefore$ A^{-1} exists.

Cofactors of matrix A are

$A_{11} = 7$ $\quad A_{12} = -19$ $\quad A_{13} = -11$

$A_{21} = 1$ $\quad A_{22} = -1$ $\quad A_{23} = -1$

$A_{31} = -3$ $\quad A_{32} = 11$ $\quad A_{33} = 7$

$$\text{adj } A = \begin{bmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{bmatrix}^T$$

$$\Rightarrow \quad \text{adj } A = \begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$= \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 49 - 5 - 36 \\ -133 + 5 + 132 \\ -77 + 5 + 84 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$\therefore$ Required values are, $x = 2$, $y = 1$, $z = 3$.

Ans.

OR

Using elementary operations, find the inverse of the following matrix :

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Solution : Let $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

We know that,

$$A = IA$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to 2R_1 + R_3$, we get

$$\begin{bmatrix} 1 & 3 & 5 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - R_1$, we get

$$\begin{bmatrix} 1 & 3 & 5 \\ 0 & -1 & -2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 + 3R_2$ and $R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & -2 \\ 0 & -8 & -14 \end{bmatrix} = \begin{bmatrix} -4 & 3 & -2 \\ -2 & 1 & -1 \\ -6 & 0 & -2 \end{bmatrix} A$$

Applying $R_2 \to -R_2$, we get

$$\begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & -8 & -14\end{bmatrix} = \begin{bmatrix}-4 & 3 & -2\\2 & -1 & 1\\-6 & 0 & -2\end{bmatrix}A$$

Applying $R_3 \to R_3 + 8R_2$, we get

$$\begin{bmatrix}1 & 0 & -1\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix} = \begin{bmatrix}-4 & 3 & -2\\2 & -1 & 1\\10 & -8 & 6\end{bmatrix}A$$

Applying $R_1 \to R_1 + \frac{1}{2}R_3$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 2\\0 & 0 & 2\end{bmatrix} = \begin{bmatrix}1 & -1 & 1\\2 & -1 & 1\\10 & -8 & 6\end{bmatrix}A$$

Applying $R_2 \to R_2 - R_3$ and $R_3 \to \frac{1}{2}R_3$, we get

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1 & 1\\-8 & 7 & -5\\5 & -4 & 3\end{bmatrix}A$$

$$I = \begin{bmatrix}1 & -1 & 1\\-8 & 7 & -5\\5 & -4 & 3\end{bmatrix}A$$

$$\therefore \quad A^{-1} = \begin{bmatrix}1 & -1 & 1\\-8 & 7 & -5\\5 & -4 & 3\end{bmatrix}$$ **Ans.**

Mathematics 2012 (Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

> **Note : Except for the following questions, all the remaining questions have been asked in previous set.**

SECTION — A

9. Find the sum of the following vectors :
$\vec{a} = \hat{i} - 2\hat{j}, \vec{b} = 2\hat{i} - 3\hat{j}, \vec{c} = 2\hat{i} + 3\hat{k}$. [1]

Solution : The given vectors are

$$\vec{a} = \hat{i} - 2\hat{j}, \vec{b} = 2\hat{i} - 3\hat{j}, \vec{c} = 2\hat{i} + 3\hat{k}$$

$$\therefore \vec{a} + \vec{b} + \vec{c} = (\hat{i} - 2\hat{j}) + (2\hat{i} - 3\hat{j}) + (2\hat{i} + 3\hat{k})$$

$$= 5\hat{i} - 5\hat{j} + 3\hat{k}.$$ **Ans.**

10. If $\Delta = \begin{vmatrix}5 & 3 & 8\\2 & 0 & 1\\1 & 2 & 3\end{vmatrix}$, write the cofactor of the element a_{32}. [1]

Solution : Given, $\Delta = \begin{vmatrix}5 & 3 & 8\\2 & 0 & 1\\1 & 2 & 3\end{vmatrix}$ and $a_{32} = 2$

Minor of a_{32}

$$= \begin{vmatrix}5 & 8\\2 & 1\end{vmatrix} = 5 - 16 = -11$$

$\therefore$ Cofactor of $a_{32} = (-1)^{3+2}(-11) = 11.$ **Ans.**

SECTION — B

19. Using properties of determinants, prove the following :

$$\begin{vmatrix}1 & 1 & 1\\a & b & c\\a^3 & b^3 & c^3\end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$ [4]

Solution : L.H.S.

$$= \begin{vmatrix}1 & 1 & 1\\a & b & c\\a^3 & b^3 & c^3\end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$ and $C_2 \to C_2 - C_3$, we get

$$= \begin{vmatrix}0 & 0 & 1\\a-b & b-c & c\\a^3 - b^3 & b^3 - c^3 & c^3\end{vmatrix}$$

$$= \begin{vmatrix}0 & 0 & 1\\a-b & b-c & c\\(a-b)(a^2+ab+b^2) & (b-c)(b^2+bc+c^2) & c^3\end{vmatrix}$$

Taking $(a-b)$ and $(b-c)$ common from C_1 and C_2 respectively, we get

$$= (a-b)(b-c)\begin{vmatrix}0 & 0 & 1\\1 & 1 & c\\(a^2+ab+b^2) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

Expanding along R_1, we get,

$(a-b).(b-c)\,[(b^2+bc+c^2)-(a^2+ab+b^2)]$

$=(a-b).(b-c)\,[(c^2-a^2)+(bc-ab)]$

$=(a-b).(b-c)\,[(c-a)(c+a)+b(c-a)]$

$=(a-b)(b-c)(c-a)(c+a+b)$

$=(a-b)(b-c)(c-a)(a+b+c)=$ R.H.S.

Hence Proved.

20. If $y=3\cos(\log x)+4\sin(\log x)$, show that

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0 \qquad [4]$$

Solution : Given, $y=3\cos(\log x)+4\sin(\log x)$ ...(i)

Differentiating w.r.t. x, we get

$$\frac{dy}{dx}=-3\sin(\log x).\frac{d}{dx}(\log x)+4\cos(\log x).\frac{d}{dx}(\log x)$$

$$=-3\sin(\log x).\frac{1}{x}+4\cos(\log x).\frac{1}{x}$$

$$\Rightarrow \quad x.\frac{dy}{dx}=-3\sin(\log x)+4\cos(\log x)$$

Again differentiating w.r.t. x, we get

$$x.\frac{d^2y}{dx^2}+1\frac{dy}{dx}=-3\cos(\log x).\frac{1}{x}-4\sin(\log x).\frac{1}{x}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=-[3\cos(\log x)+4\sin(\log x)]$$

$$=-y \qquad \text{[using (i)]}$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$

Hence Proved.

21. Find the equation of the line passing through the point $(-1, 3, -2)$ and perpendicular to the lines

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3} \text{ and } \frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+1}{5} \qquad [4]$$

Solution : Let a, b, c be the direction ratios of the line which is perpendicular to the lines

$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and $\frac{x+2}{-3}=\frac{y-1}{2}=\frac{z+1}{5}$, then

$$a+2b+3c=0 \qquad ...(i)$$

and $$-3a+2b+5c=0 \qquad ...(ii)$$

Solving equation (i) and (ii), we get

$$\frac{a}{10-6}=\frac{b}{-9-5}=\frac{c}{2+6}$$

$$\Rightarrow \quad \frac{a}{4}=\frac{b}{-14}=\frac{c}{8}$$

$$\Rightarrow \quad \frac{a}{2}=\frac{b}{-7}=\frac{c}{4}$$

$\therefore$ Equation of the required line passing through $(-1, 3, -2)$ having d.r.s. $2, -7, 4$ is

$$\frac{x+1}{2}=\frac{y-3}{-7}=\frac{z+2}{4} \qquad \textbf{Ans.}$$

22. Find the particular solution of the following differential equation;

$$(x+1)\frac{dy}{dx}=2e^{-y}-1;\ y=0 \text{ when } x=0. \qquad [4]$$

Solution : The given differential equation is

$$(x+1)\frac{dy}{dx}=2e^{-y}-1 \qquad ...(i)$$

Separate the given differential equation, we get

$$\frac{dy}{2e^{-y}-1}=\frac{dx}{x+1}$$

$$\Rightarrow \quad \frac{e^y}{2-e^y}dy=\frac{dx}{x+1}$$

On integrating, we get

$$\int\frac{e^y}{2-e^y}dy=\int\frac{dx}{x+1}+C$$

$$\Rightarrow \quad -\log(2-e^y)=\log(x+1)+C \qquad ...(ii)$$

Putting $y=0$, when $x=0$ in (ii), we get

$$-\log(2-1)=\log(0+1)+C$$

$$\Rightarrow \quad 0=0+C \Rightarrow C=0$$

$\therefore$ Equation (ii) becomes

$$-\log(2-e^y)=\log(x+1)$$

$$\Rightarrow \quad \log(x+1)+\log(2-e^y)=0$$

$$\Rightarrow \quad \log(x+1)(2-e^y)=0$$

$$\therefore \quad (x+1)(2-e^y)=e^0=1. \qquad \textbf{Ans.}$$

SECTION — C

28. A girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin two times and notes the number of heads obtained. If she obtained exactly two heads, what is the probability that she threw 1, 2, 3 or 4 with the die ? [6]

Solution : Let A_1 be even that the girl gets 5 or 6 and hence tosses a coin 3-times.

A_2 be the even that girl gets 1, 2, 3 or 4 and hence tosses a coin 2-times.

and A be even that the girl gets exactly two heads.

Now $P(A_1) = P(5 \text{ or } 6) = P(5) + P(6) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$P(A_2) = P(1, 2, 3 \text{ or } 4)$

$= P(1) + P(2) + P(3) + P(4)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$

and $P(A/A_1) = 3/8$

Here $n = 3$; sample space = {HHH, HTH, HHT, THH, HTT, THT, TTH, TTT}

and $P(A/A_2) = \frac{1}{4}$

Here $n = 2$;

Sample space = {HH, HT, TH, TT}

$\therefore$ By Bayes' theorem

$$P(A_2/A) = \frac{P(A_2).P(A/A_2)}{P(A_1).P(A/A_1) + P(A_2).P(A/A_2)}$$

$$= \frac{\frac{2}{3} \times \frac{1}{4}}{\frac{1}{3} \times \frac{3}{8} + \frac{2}{3} \times \frac{1}{4}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{8} + \frac{1}{6}} = \frac{4}{7}$$ **Ans.**

29. Using the method of integration, find the area of the region bounded by the following lines $3x - y - 3 = 0$, $2x + y - 12 = 0$, $x - 2y - 1 = 0$. **[6]**

Solution : The given lines are

$3x - y - 3 = 0 \Rightarrow y = 3x - 3$ ...(i)

$2x + y - 12 = 0 \Rightarrow y = 12 - 2x$ ...(ii)

$x - 2y - 1 = 0 \Rightarrow y = \frac{x-1}{2}$ ...(iii)

Table for the line (i),

x	1	3
y	0	6

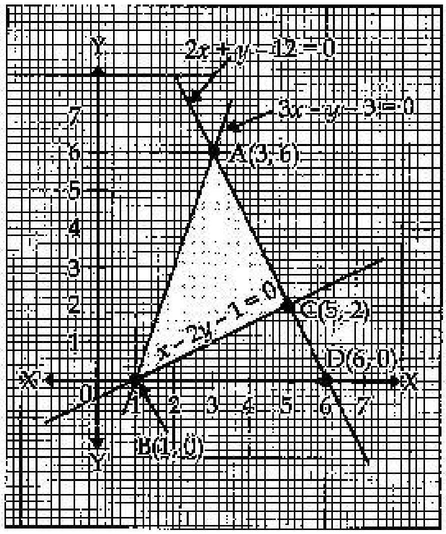
Table for the line (ii),

x	6	3	5
y	0	6	2

Table for the line (iii),

x	1	5
y	0	2

Now we draw these lines (i), (ii) and (iii).



These lines intersect at A(3, 6), B(1, 0) and C(5, 2)

$\therefore$ Area of ΔABC = Area under the line AB

+ Area under the line AC

− Area under the line BC

$$= \int_1^3 [\text{line (i)}]\,dx + \int_3^5 [\text{line (ii)}]\,dx - \int_1^5 [\text{line (iii)}]\,dx$$

$$= \int_1^3 (3x - 3)\,dx + \int_3^5 (12 - 2x)\,dx - \int_1^5 \left(\frac{x-1}{2}\right) dx$$

$$= \left[\frac{3}{2}x^2 - 3x\right]_1^3 + [12x - x^2]_3^5 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^5$$

$$= \left\{\frac{27}{2} - 9 - \left(\frac{3}{2} - 3\right)\right\} + \{60 - 25 - (36 - 9)\}$$

$$- \frac{1}{2}\left\{\frac{25}{2} - 5 - \left(\frac{1}{2} - 1\right)\right\}$$

$$= \frac{9}{2} + \frac{3}{2} + 35 - 27 - \frac{1}{2}\left(\frac{25 - 10 + 1}{2}\right)$$

$= 6 + 8 - 4 = 10$ sq. units. **Ans.**

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Time allowed : 3 hours

Maximum marks : 100

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION — A

9. Find the sum of the following vectors :

$\vec{a} = \hat{i} - 3\hat{k}, \vec{b} = 2\hat{j} - \hat{k}, \vec{c} = 2\hat{i} - 3\hat{j} + 2\hat{k}.$ **[1]**

Solution : The given vectors are

$\vec{a} = \hat{i} - 3\hat{k}, \vec{b} = 2\hat{j} - \hat{k}, \vec{c} = 2\hat{i} - 3\hat{j} + 2\hat{k}$

$$\therefore \vec{a} + \vec{b} + \vec{c} = (\hat{i} - 3\hat{k}) + (2\hat{j} - \hat{k}) + (2\hat{i} - 3\hat{j} + 2\hat{k})$$

$$= 3\hat{i} - \hat{j} - 2\hat{k}.$$ **Ans.**

10. If $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 5 & 3 & 8 \end{vmatrix}$, write the minor of the element a_{22}. **[1]**

Solution : Given, $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 5 & 3 & 8 \end{vmatrix}$

$$\therefore \quad \text{Minor of } a_{22} = \begin{vmatrix} 1 & 3 \\ 5 & 8 \end{vmatrix}$$

$$= 8 - 15 = -7.$$ **Ans.**

SECTION–B

19. Using properties of determinants, prove the following :

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = ab + bc + ca + abc.$$ **[4]**

Solution : L.H.S. = $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$= \begin{vmatrix} a & 0 & 1 \\ 0 & b & 1 \\ -c & -c & 1+c \end{vmatrix}$$

Applying $C_1 \to C_1 - a.C_3$, we get

$$= \begin{vmatrix} 0 & 0 & 1 \\ -a & b & 1 \\ -c-a-ac & -c & 1+c \end{vmatrix}$$

Expanding along R_1, we get

$ac - b(-c - a - ac)$

$= ac + bc + ba + bac$

$= ab + bc + ca + abc = \text{R.H.S.}$ **Hence Proved.**

20. If $y = \sin^{-1} x$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$ **[4]**

Solution : Given, $y = \sin^{-1} x$,

Differentiating w.r. to x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad \sqrt{1-x^2}.\frac{dy}{dx} = 1$$

Again differentiating w.r. t. x, we get

$$\sqrt{1-x^2}.\frac{d^2y}{dx^2} + \frac{1}{2}.\frac{1}{\sqrt{1-x^2}}.(-2x)\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \sqrt{1-x^2}.\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}.\frac{dy}{dx} = 0$$

Multiplying by $\sqrt{1-x^2}$ on both sides, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x.\frac{dy}{dx} = 0.$$

Hence Proved.

21. Find the particular solution of the following differential equation :

$xy\frac{dy}{dx} = (x+2)(y+2)$; $y = -1$ when $x = 1$. **[4]**

Solution : The given differential equation is

$$xy\frac{dy}{dx} = (x+2)(y+2) \quad \text{...(i)}$$

Separate the given differential equation, we get

$$\frac{y}{y+2}dy = \frac{x+2}{x}dx$$

On integrating, we get

$$\int \frac{y}{y+2}dy = \int \frac{x+2}{x}dx + C$$

$$\Rightarrow \int\left[1-\frac{2}{y+2}\right]dy = \int\left[1+\frac{2}{x}\right]dx + C$$

$$\Rightarrow \quad y-2\log(y+2) = x+2\log x + C \qquad ...(ii)$$

Putting $y=-1$, when $x=1$ in equation (ii), we get

$-1-2\log 1 = 1+2\log 1 + C$

$\Rightarrow -1-0 = 1+0+C \Rightarrow C = -2$

$\therefore$ From equation (ii),

$$y-2\log(y+2) = x+2\log x - 2$$

$$\Rightarrow \quad y-x+2 = 2[\log x + \log(y+2)]$$

$$\Rightarrow \quad y-x+2 = 2\log\{x(y+2)\}$$

This is the required particular solution of the given differential equation. **Ans.**

22. Find the equation of a line passing through the point P(2, – 1, 3) and perpendicular to the lines

$\vec{r} = (\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$ **and**

$\vec{r} = (2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$. **[4]**

Solution : The given lines are

$$\vec{r} = (\hat{i}+\hat{j}-\hat{k})+\lambda(2\hat{i}-2\hat{j}+\hat{k})$$

$$\vec{r} = (2\hat{i}-\hat{j}-3\hat{k})+\mu(\hat{i}+2\hat{j}+2\hat{k})$$

The required line is $\perp$ to both these lines which are parallel to the vectors

$\vec{b_1} = 2\hat{i}-2\hat{j}+\hat{k}$ and $\vec{b_2} = \hat{i}+2\hat{j}+2\hat{k}$, respectively

$\Rightarrow$ The required line is parallel to the vector

$$\vec{b} = \vec{b_1}\times\vec{b_2}$$

Now, $\vec{b} = \vec{b_1}\times\vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -6\hat{i}-3\hat{j}+6\hat{k}$

Now, to find the line passing through (2, – 1, 3) which is parallel to vector $\vec{b}$.

$$\therefore \quad \vec{r} = (2\hat{i}-\hat{j}+3\hat{k})+t(-6\hat{i}-3\hat{j}+6\hat{k})$$

$$= (2\hat{i}-\hat{j}+3\hat{k})-3t(2\hat{i}+\hat{j}-2\hat{k})$$

$$= (2\hat{i}-\hat{j}+3\hat{k})+\lambda(2\hat{i}+\hat{j}-2\hat{k})$$

where $\lambda = -3t$. **Ans.**

SECTION–C

28. Bag I contains 3 red and 4 black balls and Bag II contains 4 red and 5 black balls. Two balls are transferred at random from Bag I to Bag II and then a ball is drawn from Bag II. The ball so drawn is found to be red in colour. Find the probability that the transferred balls were both black. [6]

Solution : Let,

E_1 = Both transferred balls from bag I to bag II are red.

E_2 = Both transferred balls from bag I to bag II are black.

E_3 = Out of transferred balls one is black and other is red.

A = Drawing a red ball from bag II

$$P(E_1) = \frac{^3C_2}{^7C_2} = \frac{3!}{2!\times 1!}\times\frac{2!\times 5!}{7!} = \frac{1}{7}$$

$$P(E_2) = \frac{^4C_2}{^7C_2} = \frac{4!}{2!\times 1!}\times\frac{2!\times 5!}{7!} = \frac{2}{7}$$

$$P(E_3) = \frac{^3C_1\times {^4C_1}}{^7C_2} = \frac{3\times 4}{7!}\times\frac{2!\times 5!}{7!} = \frac{4}{7}$$

$$P(A/E_1) = \frac{6}{11}, \quad P(A/E_2) = \frac{4}{11}, \quad P(A/E_3) = \frac{5}{11}$$

Required probability

$P(E_2/A)$ =

$$\frac{P(E_2).P(A/E_2)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)+P(E_3).P(A/E_3)}$$

$$= \frac{\frac{2}{7}\times\frac{4}{11}}{\frac{1}{7}\times\frac{6}{11}+\frac{2}{7}\times\frac{4}{11}+\frac{4}{7}\times\frac{5}{11}}$$

$$= \frac{\frac{8}{77}}{\frac{6}{77}+\frac{7}{77}+\frac{20}{77}}$$

$$= \frac{4}{17}$$ **Ans.**

29. Using the method of integration, find the area of the region bounded by the following lines $5x - 2y - 10 = 0$, $x + y - 9 = 0$, $2x - 5y - 4 = 0$. [6]

Solution : The given lines are

$$5x - 2y - 10 = 0 \Rightarrow y = \frac{5x-10}{2} \quad \text{...(i)}$$

$$x + y - 9 = 0 \Rightarrow y = 9 - x \quad \text{...(ii)}$$

$$2x - 5y - 4 = 0 \Rightarrow y = \frac{2x-4}{5} \quad \text{...(iii)}$$

Table for the line (i),

x	2	4
y	0	5

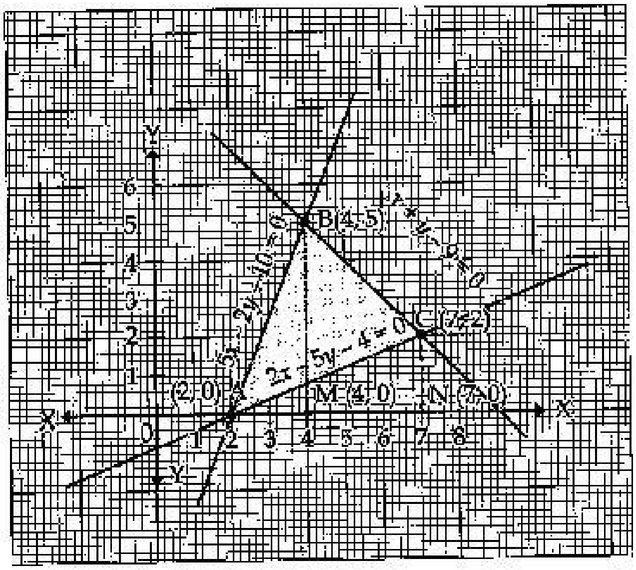
Table for the line (ii),

x	7	4
y	2	5

Table for the line (iii),

x	2	7
y	0	2

Now we draw these lines (i), (ii) and (iii),



These lines intersect at A(2, 0), B(4, 5) and C(7, 2).

∴ Area of Δ ABC = Area AMB + Area BMNC − Area ANC

$$= \int_2^4 [\text{line (i)}]\,dx + \int_4^7 [\text{line (ii)}]\,dx - \int_2^7 [\text{line (iii)}]\,dx$$

$$= \int_2^4 \frac{5x-10}{2}\,dx + \int_4^7 (9-x)\,dx - \int_2^7 \frac{2x-4}{5}\,dx$$

$$= \frac{1}{2}\left[\frac{5x^2}{2} - 10x\right]_2^4 + \left[9x - \frac{x^2}{2}\right]_4^7 - \frac{1}{5}\left[x^2 - 4x\right]_2^7$$

$$= \frac{1}{2}[40 - 40 - (10 - 20)]$$

$$+ \left[63 - \frac{49}{2} - (36 - 8)\right] - \frac{1}{5}[49 - 28 - (4 - 8)]$$

$$= 5 + 35 - \frac{49}{2} - 5 = \frac{21}{2} \text{ sq. units.}$$ **Ans.**

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