Mathematics 2011 (Outside Delhi) Set I

Time allowed : 3 Hours **Maximum marks : 100**

SECTION – A

1. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. State whether f is one-one or not. [1]

Solution : Given,

$$A = \{1, 2, 3\},$$
$$B = \{4, 5, 6, 7\}$$
$$f = \{(1, 4), (2, 5), (3, 6)\}$$

$f : A \to B$ is defined as

$\therefore\ f(1) = 4,\ f(2) = 5,\ f(3) = 6.$

Different points of the domain have different f-image in the range.

$\therefore$ f is one-one. **Ans.**

2. What is the principal value of

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)? \quad [1]$$

Solution : Given,

$$\cos^{-1}\left(\cos\frac{2\pi}{3}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$$

We know that principal value branch of $\cos^{-1}$ is $[0, \pi]$ and $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore$ The principal value of

$$\cos^{-1}\left[\cos\frac{2\pi}{3}\right]+\sin^{-1}\left[\sin\frac{2\pi}{3}\right]$$
$$= \frac{2\pi}{3}+\sin^{-1}\left[\sin\left(\pi-\frac{2\pi}{3}\right)\right]$$
$$= \frac{2\pi}{3}+\sin^{-1}\left(\sin\frac{\pi}{3}\right)$$
$$= \frac{2\pi}{3}+\frac{\pi}{3} = \pi. \quad \textbf{Ans.}$$

3. Evaluate : $\begin{vmatrix}\cos 15^\circ & \sin 15^\circ\\ \sin 75^\circ & \cos 75^\circ\end{vmatrix}$. **[1]**

Solution : $\begin{vmatrix}\cos 15^\circ & \sin 15^\circ\\ \sin 75^\circ & \cos 75^\circ\end{vmatrix}$

$$= \cos 75^\circ.\cos 15^\circ - \sin 75^\circ.\sin 15^\circ$$
$$= \cos(75^\circ + 15^\circ)$$

$[\because \cos(A + B) = \cos A \cos B - \sin A \sin B]$

$$= \cos 90^\circ$$
$$= 0. \quad \textbf{Ans.}$$

4. If A = $\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix}$ **write A^{-1} in terms of A. [1]**

Solution : Here, $|A| = 2(-2) - 5(3)$

$$= -4 - 15 = -19 \neq 0$$

$\therefore$ A^{-1} exists

and $\quad \text{adj } A = \begin{bmatrix}-2 & -3\\ -5 & 2\end{bmatrix}$

$$\therefore \quad A^{-1} = \frac{1}{|A|}.\text{adj } A$$
$$= \frac{-1}{19}\begin{bmatrix}-2 & -3\\ -5 & 2\end{bmatrix}$$
$$= \frac{1}{19}\begin{bmatrix}2 & 3\\ 5 & -2\end{bmatrix} = \frac{1}{19}A. \quad \textbf{Ans.}$$

5. If a matrix has 5 elements, write all possible orders it can have. [1]

Solution : Since a matrix of order $m \times n$ has mn elements therefore, to find all possible orders of a matrix with 5 elements, we have to fill all possible ordered pairs (m, n) of positive integers whose product is 5. Hence possible orders are 1×5 and 5×1. **Ans.**

6. Evaluate : $\int (ax+b)^3\, dx$ **[1]**

Solution : Let, $I = \int (ax+b)^3.dx$

Let $\quad t = ax + b$

Differentiating w.r.t. x, we get

$$\frac{dt}{dx} = a - 0$$
$$\Rightarrow \quad dx = \frac{dt}{a}$$
$$I = \int t^3.\frac{dt}{a}$$
$$\therefore \quad I = \frac{1}{a}\int t^3.dt$$
$$= \int t^3.dt\frac{1}{a}$$
$$= \frac{1}{a}\left(\frac{t^4}{4}\right)+C$$
$$= \frac{1}{4a}t^4 + C$$
$$= \frac{1}{4a}(ax+b)^4 + C \quad \textbf{Ans.}$$

7. Evaluate : $\int \frac{dx}{\sqrt{1-x^2}}$. **[1]**

Solution : Let $\quad I = \int \frac{dx}{\sqrt{1-x^2}}$.

$$\left[\because \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)+C\right]$$
$$= \sin^{-1} x + C. \quad \textbf{Ans.}$$

8. Write the direction-cosines of the line joining the points (1, 0, 0) and (0, 1, 1). **[1]**

Solution : The d.r's of line joining points (1, 0, 0) and (0, 1, 1) are 0 – 1, 1 – 0, 1 – 0

i.e. –1, 1, 1

$$\therefore \text{D.C's are } \frac{-1}{\sqrt{1^2+1^2+1^2}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$
$$= \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}. \quad \textbf{Ans.}$$

9. Write the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$. **[1]**

Solution : Let $\vec{a} = \hat{i}-\hat{j}$ and $\vec{b} = \hat{i}+\hat{j}$.

Now, projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|} = \frac{(\hat{i}-\hat{j}).(\hat{i}+\hat{j})}{\sqrt{1^2+1^2}} = \frac{1-1}{\sqrt{2}} = \frac{0}{\sqrt{2}} = 0. \quad \textbf{Ans.}$$

10. Write the vector equation of a line given by $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$. **[1]**

Solution : The given line is

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2} \quad ...(i)$$

The equation of line (i) comparing with

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

We have $x_1 = 5, y_1 = -4, z_1 = 6$ and $a = 3, b = 7, c = 2$

Fixed point vector,

$$\vec{a} = x_1\hat{i}+y_1\hat{j}+z_1\hat{k} = 5\hat{i}-4\hat{j}+6\hat{k}$$

Direction vector,

$$\vec{b} = a\hat{i}+b\hat{j}+c\hat{k} = 3\hat{i}+7\hat{j}+2\hat{k}$$

$\therefore$ Vector equation of the given line is

$$\vec{r} = \vec{a} + \lambda\vec{b}$$
$$\vec{r} = (5\hat{i}-4\hat{j}+6\hat{k}) + \lambda(3\hat{i}+7\hat{j}+2\hat{k}).$$

Ans.

SECTION – B

11. Let $f : R \to R$ be defined as $f(x) = 10x + 7$. Find the function $g : R \to R$ such that $gof = fog = I_R$. **[4]**

Solution : It is given that $f : R \to R$ is defined as

$$f(x) = 10x + 7$$

One-one

Let $f(x) = f(y)$, where $x, y \in R$

$\Rightarrow$ $10x + 7 = 10y + 7$

$\Rightarrow$ $x = y$

$\therefore f$ is a one-one function.

Onto :

For $y \in R$, let $y = 10x + 7$

$$\therefore \quad x = \frac{y-7}{10} \in R$$

Therefore, for any $y \in R$, there exists $x = \frac{y-7}{10} \in R$ such that

$$f(x) = f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right)+7 = y-7+7 = y$$

$\therefore f$ is onto.

Therefore, f is one-one and onto.

Thus, f is an invertible function.

Let us define $f : R \to R$ as $g(y) = \frac{y-7}{10}$

Now we have,

$$gof(x) = g(f(x)) = g(10x+7)$$
$$= \frac{10x+7-7}{10} = x$$
$$fog(y) = f(g(y))$$
$$\Rightarrow \quad f\left(\frac{y-7}{10}\right) = 10\left(\frac{y-7}{10}\right)+7 = y-7+7 = y$$

$\therefore$ $gof = I_R$ and $fog = I_R$

Hence, the required function on $g : R \to R$ is defined as

$$g(y) = \frac{y-7}{10} \quad \textbf{Ans.}$$

OR

A binary operation * on the set {0, 1, 2, 3, 4, 5} is defined as :

$$a*b = \begin{cases} a+b, & \text{if } a+b<6 \\ a+b-6, & \text{if } a+b \geq 6 \end{cases}$$

Show that zero is the identity for this operation and each element 'a' of the set is invertible with $6-a$, being the inverse of 'a'.**

12. Prove that : $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x, -\frac{1}{\sqrt{2}} \leq x \leq 1$ **[4]**

Solution : L.H.S.

$$= \tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

**Answer is not given due to the change in present syllabus

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$ and $0 \le \theta \le \frac{3\pi}{4}$.

$$\therefore \quad \tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$$

$$= \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$$

[$\because 1+\cos\theta = 2\cos^2(\theta/2)$ and $1-\cos\theta = 2\sin^2(\theta/2)$]

$$= \tan^{-1}\left[\frac{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{2}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)}{\sqrt{2}\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\right)}\right]$$

Inside the bracket divide numerator and denominator by $\cos\frac{\theta}{2}$, we get

$$= \tan^{-1}\left[\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}.\tan\frac{\theta}{2}}\right]$$

$$= \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$

$$= \frac{\pi}{4}-\frac{\theta}{2}$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1} x = \text{R.H.S.}$$

Hence Proved.

13. Using properties of determinants, solve the following for x :

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0. \qquad [4]$$

Solution : L.H.S. $= \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$

Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$, we get

$$= \begin{vmatrix} 2 & 6 & 12 \\ 4 & 18 & 48 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$$

Taking 2 common from R_1 and R_2

$$= 2\times 2\begin{vmatrix} 1 & 3 & 6 \\ 2 & 9 & 24 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix}$$

Applying $C_2 \to C_2 - 3C_1$; $C_3 \to C_3 - 2C_2$, we get

$$= 4\begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 6 \\ x-8 & -x-3 & -x-10 \end{vmatrix}$$

Expanding along C_1, we get

$$= 4(-3x - 30 + 6x + 18)$$
$$= 4[3x - 12] = 0$$
$$\Rightarrow \quad 3x = 12$$
$$\Rightarrow \quad x = 4.$$

Ans.

14. Find the relationship between 'a' and 'b' so that the function 'f' defined by :

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$

is continuous at $x = 3$. **[4]**

Solution : $\because f(x)$ is continuous at $x = 3$,

$$\therefore \quad f(3) = \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x)$$

$$\Rightarrow \quad \lim_{x\to 3^-}(ax+1) = \lim_{x\to 3^+}(bx+3)$$

$$\Rightarrow \quad 3a+1 = 3b+3$$

$$\Rightarrow \quad 3a = 3b+2$$

$$\Rightarrow \quad a = b+\frac{2}{3}.$$

Ans.

OR

If $x^y = e^{x-y}$, show that $\frac{dy}{dx} = \frac{\log x}{\{\log(xe)\}^2}$.

Solution : We have,

$$x^y = e^{x-y}$$

Taking log on both sides, we get

$$\log x^y = \log e^{x-y}$$
$$y\log x = (x-y)\log e = x-y$$

[$\because \log e = 1$]

$$\Rightarrow \quad y\log x + y = x$$
$$\Rightarrow \quad y(1+\log x) = x$$
$$\therefore \quad y = \frac{x}{(1+\log x)}$$

Differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(1+\log x)\frac{d}{dx}x - x\cdot\frac{d}{dx}(1+\log x)}{(1+\log x)^2}$$

$$= \frac{(1+\log x)1 - x\left(0+\frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2}$$

$$\therefore \quad \frac{dy}{dx} = \frac{\log x}{(1+\log x)^2} = \frac{\log x}{(\log e + \log x)^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\log x}{[\log(xe)]^2}.$$ **Hence Proved.**

15. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ is an increasing function in $\left[0, \frac{\pi}{2}\right]$. [4]

Solution : We have,

$$f(\theta) = \frac{4\sin\theta}{2+\cos\theta} - \theta$$

Differentiating w.r t. θ, we get

$$\therefore \quad f'(\theta) = \frac{(2+\cos\theta)(4\cos\theta) - 4\sin\theta(0-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4}{(2+\cos\theta)^2} - 1$$

$$[\because \sin^2\theta + \cos^2\theta = 1]$$

$$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - 4 - \cos^2\theta - 4\cos\theta}{(2+\cos\theta)^2}$$

$$= \frac{4\cos\theta - \cos^2\theta}{(2+\cos\theta)^2}$$

$$= \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

For all $\theta \in \left[0, \frac{\pi}{2}\right]$, $\frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2} \geq 0$, as $\cos\theta$ is +ve

Hence, $f(\theta)$ is increasing in $\left[0, \frac{\pi}{2}\right]$.

Hence Proved.

OR

If the radius of a sphere is measured as 9 cm with an error of 0.03 cm, then find the approximate error in calculating its surface area.

Solution : Let r be the radius of sphere and Δr be the error in measuring the radius.

Then $r = 9$ cm, $\Delta r = 0.03$ cm.

Now surface area S of the sphere is

$$S = 4\pi r^2$$

Differentiating w.r. to r, we get

$$\frac{dS}{dr} = 8\pi r$$

$$\therefore \quad \Delta S \approx dS = \frac{dS}{dr}\Delta r$$

$$= 8\pi r . \Delta r$$

$$= 8\pi \times 9 \times 0.03$$

$$= 2.16\,\pi \text{ cm}^2.$$

This is the approximate error in calculating surface area. **Ans.**

16. If $x = \tan\left(\frac{1}{a}\log y\right)$, show that

$$(1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0.$$ [4]

Solution : Given,

$$x = \tan\left(\frac{1}{a}\log y\right)$$

$$\Rightarrow \quad \tan^{-1} x = \frac{1}{a}\log y$$

$$\Rightarrow \quad a\tan^{-1} x = \log y$$

Differentiating both sides w.r.t. x, we get

$$a\frac{1}{1+x^2} = \frac{1}{y}\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{dy}{dx} = ay$$

Again differentiating both sides w.r.t. x, we get

$$(1+x^2)\frac{d^2y}{dx^2} + \frac{dy}{dx}.2x = a\frac{dy}{dx}$$

$$\Rightarrow \quad (1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - a\frac{dy}{dx} = 0$$

$$\therefore \quad (1+x^2)\frac{d^2y}{dx^2} + (2x-a)\frac{dy}{dx} = 0.$$

Hence Proved.

17. Evaluate : $\int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}dx$. [4]

Solution : Let $I = \int_0^{\pi/2} \frac{x+\sin x}{1+\cos x}dx$.

$$= \int_0^{\pi/2} \frac{x + 2\sin\frac{x}{2}.\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}dx$$

$$\left[\because \sin x = 2\sin\frac{x}{2}.\cos\frac{x}{2} \text{ and } \cos x = 2\cos^2\frac{x}{2} - 1\right]$$

$$= \frac{1}{2}\int_0^{\pi/2} x.\sec^2\frac{x}{2}dx + \int_0^{\pi/2}\tan\frac{x}{2}dx$$

$$= \frac{1}{2}\left[\left\{x.\frac{\tan\frac{x}{2}}{\frac{1}{2}}\right\}_0^{\pi/2} - \int_0^{\pi/2}\frac{\tan\frac{x}{2}}{\frac{1}{2}}.dx\right]$$

$$+\int_0^{\pi/2}\tan\frac{x}{2}.dx$$

$$= \left[x.\tan\frac{x}{2}\right]_0^{\pi/2}$$

$$-\int_0^{\pi/2}\tan\frac{x}{2}.dx+\int_0^{\pi/2}\tan\frac{x}{2}.dx$$

$$= \frac{\pi}{2}.\tan\frac{\pi}{4}$$

$$= \frac{\pi}{2}\times 1 = \frac{\pi}{2}.$$ **Ans.**

18. Solve the following differential equation :

$$xdy - ydx = \sqrt{x^2+y^2}dx.$$ **[4]**

Solution : The given differential equation is

$$xdy - ydx = \sqrt{x^2+y^2}dx$$

Separate the variables, we get

$$x\,dy = (y+\sqrt{x^2+y^2})dx$$

$$\therefore \quad \frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \quad ...(i)$$

This is a homogeneous differential equation.

Putting $y = vx$

$$\Rightarrow \quad \frac{dy}{dx} = v+x\frac{dv}{dx} \quad ...(ii)$$

From equation (i) and (ii), we get

$$v+x\frac{dv}{dx} = \frac{vx+\sqrt{x^2+v^2x^2}}{x}$$

$$\Rightarrow \quad v+x\frac{dv}{dx} = \frac{vx+x\sqrt{1+v^2}}{x}$$

$$\Rightarrow \quad v+x\frac{dv}{dx} = v+\sqrt{1+v^2}$$

$$\Rightarrow \quad x\frac{dv}{dx} = v+\sqrt{1+v^2}-v = \sqrt{1+v^2}$$

$$\therefore \quad \frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

On integrating both sides, we get

$$\log\left|v+\sqrt{1+v^2}\right| = \log|x| + \log|C|$$

$$\Rightarrow \log\left|\frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}\right| = \log|Cx|$$

$$\Rightarrow \quad \frac{y+\sqrt{x^2+y^2}}{x} = Cx$$

$$\Rightarrow \quad y+\sqrt{x^2+y^2} = Cx^2.$$ **Ans.**

19. Solve the following differential equation :

$$(y+3x^2)\frac{dx}{dy} = x.$$ **[4]**

Solution : The given differential equation is

$$(y+3x^2)\frac{dx}{dy} = x$$

$$\Rightarrow \quad \frac{dx}{dy} = \frac{x}{y+3x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y+3x^2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}+3x$$

$$\Rightarrow \quad \frac{dy}{dx}+\left(-\frac{1}{x}\right)y = 3x$$

This is a linear differential equation of the type

$$\frac{dy}{dx}+Py = Q$$

Here, $P = -\frac{1}{x}$ and $Q = 3x$

Integrating factor (I.F.)

$$= e^{\int P dx}$$

$$= e^{\int \frac{-1}{x}dx}$$

$$= e^{-\log x}$$

$$= e^{\log(x)^{-1}} = \frac{1}{x}$$

Required solution is

$$y.(\text{I.F.}) = \int(Q.(\text{I.F.}))dx + C$$

$$\frac{y}{x} = \int\left(3x\frac{1}{x}\right)dx + C$$

$$\Rightarrow \quad \frac{y}{x} = \int 3\,dx + C$$

$$\Rightarrow \quad \frac{y}{x} = 3x + C$$

$$\therefore \quad y = 3x^2 + Cx.$$ **Ans.**

20. Using vectors, find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5). **[4]**

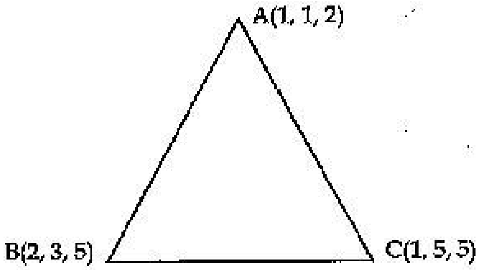
Solution : The vertices of triangle ABC are given as A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5)

Let O be the origin of triangle

$$\therefore \quad \vec{OA} = \hat{i}+\hat{j}+2\hat{k}$$

$$\vec{OB} = 2\hat{i}+3\hat{j}+5\hat{k}$$

$$\overrightarrow{OC} = \hat{i} + 5\hat{j} + 5\hat{k}$$



$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= 2\hat{i} + 3\hat{j} + 5\hat{k} - \hat{i} - \hat{j} - 2\hat{k}$$
$$= \hat{i} + 2\hat{j} + 3\hat{k}$$
$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$
$$= \hat{i} + 5\hat{j} + 5\hat{k} - \hat{i} - \hat{j} - 2\hat{k}$$
$$= 4\hat{j} + 3\hat{k}$$

Now, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$

$$= \hat{i}(6-12) - \hat{j}(3-0) + \hat{k}(4-0)$$
$$= -6\hat{i} - 3\hat{j} + 4\hat{k}$$

$\therefore \quad |\overrightarrow{AB} \times \overrightarrow{AC}| = \sqrt{(-6)^2 + (-3)^2 + (4)^2}$

$$= \sqrt{36 + 9 + 16}$$
$$= \sqrt{61}$$

$\therefore$ Area of $\Delta ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

$= \frac{1}{2}\sqrt{61}$ sq. units. **Ans.**

21. Find the shortest distance between the following lines whose vector equations are :

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \text{ and}$$
$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}. \quad [4]$$

Solution : Given equations are

$$\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) \quad ...(i)$$

and $\vec{r} = (\hat{i} - \hat{j} - \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) \quad ...(ii)$

On comparing equations (i) and (ii) with

$\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, we get

$\therefore \quad \vec{a_1} = \hat{i} - 2\hat{j} + 3\hat{k}, \vec{b_1} = -\hat{i} + \hat{j} - 2\hat{k}$

and $\vec{a_2} = \hat{i} - \hat{j} - \hat{k}, \vec{b_2} = \hat{i} + 2\hat{j} - 2\hat{k}$.

$\therefore \vec{a_2} - \vec{a_1} = \hat{i} - \hat{j} - \hat{k} - \hat{i} + 2\hat{j} - 3\hat{k} = 0\hat{i} + \hat{j} - 4\hat{k}$

Now, $\vec{b_1} \times \vec{b_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix}$

$$= \hat{i}(-2+4) - \hat{j}(2+2) + \hat{k}(-2-1)$$
$$= 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$\therefore \quad |\vec{b_1} \times \vec{b_2}| = \sqrt{(2)^2 + (-4)^2 + (-3)^2}$

$$= \sqrt{4 + 16 + 9} = \sqrt{29}$$

$\therefore$ The shortest distance between given lines is

$$\text{S.D.} = \left| \frac{(\vec{b_1} \times \vec{b_2}).(\vec{a_2} - \vec{a_1})}{|\vec{b_1} \times \vec{b_2}|} \right|$$
$$= \left| \frac{(2\hat{i} - 4\hat{j} - 3\hat{k}).(0\hat{i} + \hat{j} - 4\hat{k})}{\sqrt{29}} \right|$$
$$= \left| \frac{0 - 4 + 12}{\sqrt{29}} \right|$$

$= \frac{8}{\sqrt{29}}$ units. **Ans.**

22. A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6	7
P(X)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2$+K

Determine : **[4]**

(i) K

(ii) P(X < 3)

(iii) P(X > 6)

(iv) P(0 < X < 3).

Solution : It is known that the sum of probability distribution of variable is one.

(i) $\therefore \quad \Sigma P(X) = 1$

Therefore,

$$P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) = 1$$
$$0 + K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$
$$\Rightarrow \quad 10K^2 + 9K - 1 = 0$$
$$\Rightarrow \quad 10K^2 + 10K - K - 1 = 0$$
$$\Rightarrow \quad 10K(K+1) - 1(K+1) = 0$$
$$\Rightarrow \quad (K+1)(10K-1) = 0$$
$$\Rightarrow \quad K + 1 = 0$$

$\Rightarrow \quad K = -1$

and $\quad 10K - 1 = 0$

$\Rightarrow \quad K = 1/10$

Here, K = – 1 is not possible as the possibility of an event is never negative,

$$\therefore \quad K = \frac{1}{10}.$$

Hence, probability distribution of X is

X	0	1	2	3	4	5	6	7
P(X)	0	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{1}{100}$	$\frac{2}{100}$	$\frac{7}{100}+\frac{1}{10}$

(ii) $\quad P(X < 3) = P(0) + P(1) + P(2)$

$$= 0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$$

(iii) $\quad P(X > 6) = P(7) = \frac{7}{100} + \frac{1}{10} = \frac{17}{100}$

(iv) $\quad P(0 < X < 3) = P(1) + P(2)$

$$= \frac{1}{10} + \frac{2}{10} = \frac{3}{10}. \qquad \textbf{Ans.}$$

OR

Find the probability of throwing at most 2 sixes in 6 throws of a single die.

Solution : Here,

$$n = 6$$

$$p = P\text{ (getting 6)} = \frac{1}{6}.$$

$$\therefore q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

P (getting at most 2 sixes)

$$= P(X \le 2)$$

$$= P(0) + P(1) + P(2)$$

$$= {}^6C_0\left(\frac{5}{6}\right)^{6-0}\left(\frac{1}{6}\right)^0 + {}^6C_1\left(\frac{5}{6}\right)^{6-1}\left(\frac{1}{6}\right)^1 + {}^6C_2\left(\frac{5}{6}\right)^{6-2}\left(\frac{1}{6}\right)^2$$

$$= \left(\frac{5}{6}\right)^6 + \left(\frac{5}{6}\right)^5 + 15\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)^2$$

$$= \left(\frac{5}{6}\right)^4\left[\left(\frac{5}{6}\right)^2 + \frac{5}{6} + 15 \times \left(\frac{1}{6}\right)^2\right]$$

$$= \left(\frac{5}{6}\right)^4\left[\frac{25}{36} + \frac{5}{6} + 15 \times \frac{1}{36}\right]$$

$$= \left(\frac{5}{6}\right)^4\left[\frac{25 + 30 + 15}{36}\right]$$

$$= \left(\frac{5}{6}\right)^4 \times \frac{70}{36} = \frac{35}{18}\left(\frac{5}{6}\right)^4$$

$$= \frac{7}{3}\left(\frac{5}{6}\right)^5. \qquad \textbf{Ans.}$$

SECTION — C

23. Using matrices, solve the following system of equations:

$$4x + 3y + 2z = 60$$
$$x + 2y + 3z = 45$$
$$6x + 2y + 3z = 70. \qquad [6]$$

Solution :

$$4x + 3y + 2z = 60,$$
$$x + 2y + 3z = 45,$$
$$6x + 2y + 3z = 70.$$

The equation of system can be written in matrix form

$$AX = B \qquad ...(i)$$

$$\begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$$

Here, $A = \begin{bmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 6 & 2 & 3 \end{vmatrix}$$

$$= 4(6-6) - 3(3-18) + 2(2-12)$$

$$= 0 + 45 - 20 = 25 \ne 0$$

$\therefore A^{-1}$ exists.

Cofactors of A,

$A_{11} = (6-6) = 0$, $A_{12} = -(3-18) = 15$,

$A_{13} = (2-12) = -10$

$A_{21} = -(9-4) = -5$, $A_{22} = (12-12) = 0$

$A_{23} = -(8-18) = 10$

$A_{31} = (9-4) = 5$, $A_{32} = -(12-2) = -10$

$A_{33} = (8-3) = 5$

$$\therefore \quad \text{adj } A = \begin{bmatrix} 0 & 15 & -10 \\ -5 & 0 & 10 \\ 5 & -10 & 5 \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}$$

From (i), $\quad X = A^{-1}B$

$\therefore \quad X = \frac{1}{25}\begin{bmatrix} 0 & -5 & 5 \\ 15 & 0 & -10 \\ -10 & 10 & 5 \end{bmatrix}\begin{bmatrix} 60 \\ 45 \\ 70 \end{bmatrix}$

$= \frac{1}{25}\begin{bmatrix} 0-225+350 \\ 900+0-700 \\ -600+450+350 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 125 \\ 200 \\ 200 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$

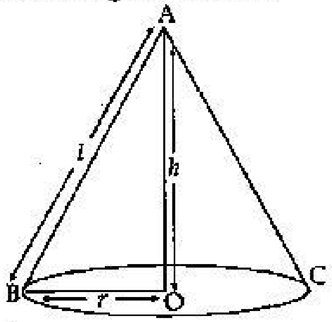
$\therefore \quad x = 5, y = 8, z = 8.$ **Ans.**

24. Show that the right circular cone of least curved surface and given volume has an altitude equal to $\sqrt{2}$ times the radius of the base. **[6]**

Solution : Let radius of cone = r

Height of cone = h

and slant height of cone = l



Curved surface area of cone

$$S = \pi r l \quad ...(i)$$

$$\therefore \quad S^2 = \pi^2 r^2 l^2$$

$$\Rightarrow \quad S^2 = \pi^2 r^2(h^2 + r^2) ...(ii)$$

$$[\because l^2 = r^2 + h^2]$$

Volume of cone

$$V = \frac{1}{3}\pi r^2 h$$

$$\Rightarrow \quad 3V = \pi r^2 h$$

$$\therefore \quad h = \frac{3V}{\pi r^2} \quad ...(iii)$$

Putting the value of h in equation (ii), we get

$$S^2 = \pi^2 r^2\left(\frac{9V^2}{\pi^2 r^4} + r^2\right)$$

$$= \pi^2 r^2\left[\frac{9V^2 + \pi^2 r^6}{\pi^2 r^4}\right]$$

$$= \left[\frac{9V^2 + \pi^2 r^6}{r^2}\right]$$

$$= \frac{9V^2}{r^2} + \pi^2 r^4$$

Let $\quad S^2 = f(r)$

$$\therefore \quad f(r) = \frac{9V^2}{r^2} + \pi^2 r^4$$

Differentiating w.r.t. r, we get

$$f'(r) = -18V^2 r^{-3} + 4\pi^2 r^3 \quad ...(iv)$$

For stationary points,

$$f'(r) = 0$$

$$\Rightarrow \quad -18V^2 r^{-3} + 4\pi^2 r^3 = 0$$

$$\Rightarrow \quad \frac{-18V^2}{r^3} = -4\pi^2 r^3$$

$$\Rightarrow \quad 4\pi^2 r^6 = 18V^2$$

$$\Rightarrow \quad r^6 = \frac{18V^2}{4\pi^2}$$

$$\Rightarrow \quad r^6 = \frac{9V^2}{2\pi^2}$$

$$\Rightarrow \quad r^3 = \frac{3V}{\sqrt{2}\pi}$$

$$\Rightarrow \quad r = \left(\frac{3V}{\sqrt{2}\pi}\right)^{1/3}$$

Again differentiating (iv), we get

$$f''(r) = 54V^2 r^{-4} + 12\pi^2 r^2$$

$$\therefore \text{ At} \quad r = \left(\frac{3V}{\sqrt{2}\pi}\right)^{1/3}$$

$$f''(r) = 54V^2\left(\frac{3V}{\sqrt{2}\pi}\right)^{-4/3} + 12\pi^2\left(\frac{3V}{\sqrt{2}\pi}\right)^2 > 0$$

$\therefore f(r)$ is minimum at

$$r = \left(\frac{3V}{\sqrt{2}\pi}\right)^{1/3}$$

$\therefore$ Surface area is minimum at

$$r^3 = \frac{3V}{\sqrt{2}\pi}$$

$$\therefore \quad V = \frac{\sqrt{2}\pi r^3}{3}$$

Putting the value of V in equation (iii), we get

$$h = \frac{\frac{3\sqrt{2}\pi r^3}{3}}{\pi r^2}$$

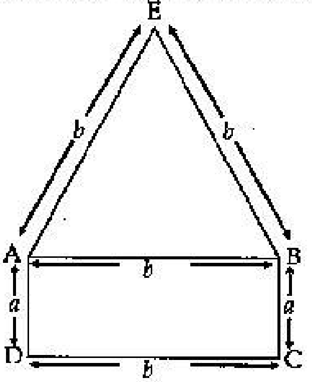
$$\Rightarrow \quad h = \sqrt{2}r$$

Hence, altitude is equal to $\sqrt{2}$ times the radius of base. **Hence Proved.**

OR

A window has the shape of a rectangle surmounted by an equilateral triangle. If the perimeter of the window is 12 m, find the dimensions of the rectangle that will produce the largest area of the window.

Solution : Let ABCD be a rectangle and let the side of an equilateral triangle be AB = b (length of the rectangle) and BC = a (width of the rectangle)



Total perimeter of the window = 12

$$\therefore \quad b + 2a + 2b = 12$$

$$\Rightarrow \quad 2a + 3b = 12$$

$$\Rightarrow \quad a = \frac{12-3b}{2} \quad \text{...(i)}$$

Now, area A of the window = Area of rectangle + Area of equilateral Δ

$$A = ab + \frac{\sqrt{3}}{4}b^2$$

$$= \left(\frac{12-3b}{2}\right)b + \frac{\sqrt{3}}{4}b^2 \quad \text{[using (i)]}$$

$$\therefore \quad A = 6b - \frac{3}{2}b^2 + \frac{\sqrt{3}}{4}b^2$$

Differentiating w.r.t. b, we get

$$\frac{dA}{db} = 6 - 3b + \frac{\sqrt{3}}{2}b$$

For maxima or minima,

$$\frac{dA}{db} = 0$$

$$\Rightarrow \quad \left(3 - \frac{\sqrt{3}}{2}\right)b = 6$$

$$\Rightarrow \quad b = \frac{12}{6-\sqrt{3}}$$

Also, $$\frac{d^2A}{db^2} = -3 + \frac{\sqrt{3}}{2} < 0$$

$\Rightarrow$ Area is maximum, when $b = \dfrac{12}{6-\sqrt{3}}$

From (i),

$$a = \frac{12-3b}{2} = 6 - \frac{3}{2}\cdot\frac{12}{6-\sqrt{3}}$$

$$a = \frac{18-6\sqrt{3}}{6-\sqrt{3}}.$$

Hence, the dimensions of the rectangle that will produce the largest area of the window are $\left(\dfrac{18-6\sqrt{3}}{6-\sqrt{3}}\right)$ and $\left(\dfrac{12}{6-\sqrt{3}}\right)$. **Ans.**

25. Evaluate : $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$. **[6]**

Solution : Let,

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$$

$$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\frac{\sqrt{\sin x}}{\sqrt{\cos x}}}$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}\,dx}{\sqrt{\cos x}+\sqrt{\sin x}} \quad \text{...(i)}$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\right]$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}}{\sqrt{\cos\left(\frac{\pi}{2}-x\right)}+\sqrt{\sin\left(\frac{\pi}{2}-x\right)}}dx$$

$$\Rightarrow \quad I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\sin x}\,dx}{\sqrt{\sin x}+\sqrt{\cos x}} \quad \text{...(ii)}$$

On adding equations (i) and (ii), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}+\sqrt{\sin x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$$

$$\Rightarrow \quad 2I = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1.dx$$

$$\Rightarrow \quad 2I = [x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$\Rightarrow \quad 2I = \left[\frac{\pi}{3} - \frac{\pi}{6}\right]$$

$$\Rightarrow \quad 2I = \left[\frac{2\pi - \pi}{6}\right]$$

$$\Rightarrow \quad 2I = \frac{\pi}{6}$$

$$\therefore \quad I = \frac{\pi}{12}$$

Ans.

OR

Evaluate : $\int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx.$

Solution : Let $I = \int \frac{6x+7}{\sqrt{(x-5)(x-4)}}dx$

$$= \int \frac{6x+7}{\sqrt{x^2-9x+20}}$$

Let $\quad 6x + 7 = A\frac{d}{dx}(x^2 - 9x + 20) + B$

$\therefore \quad 6x + 7 = A(2x - 9) + B$

Equating the coefficients of like terms from both sides, we get

$2A = 6$

$\Rightarrow \quad A = 3$

and $\quad -9A + B = 7$

$\Rightarrow \quad -9 \times 3 + B = 7$

$\therefore \quad B = 7 + 27 = 34$

$$\therefore \quad I = \int \frac{3(2x-9)}{\sqrt{x^2-9x+20}}dx + \int \frac{34}{\sqrt{x^2-9x+20}}dx$$

Putting $x^2 - 9x + 20 = t$

$\Rightarrow \quad (2x - 9)dx = dt$ in first integral, we get

$$\therefore \quad I = \int \frac{3}{\sqrt{t}}dt + 34\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2 + 20 - \frac{81}{4}}}$$

$$= 3\int t^{-1/2}dt + 34\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2 - \frac{1}{4}}}$$

$$= 3\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + 34\int \frac{dx}{\sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$= 6\sqrt{t} + 34\log\left|\left(x-\frac{9}{2}\right) + \sqrt{\left(x-\frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}\right| + C$$

$$= 6\sqrt{x^2-9x+20} + 34\log\left|\left(x-\frac{9}{2}\right) + \sqrt{x^2-9x+20}\right| + C.$$

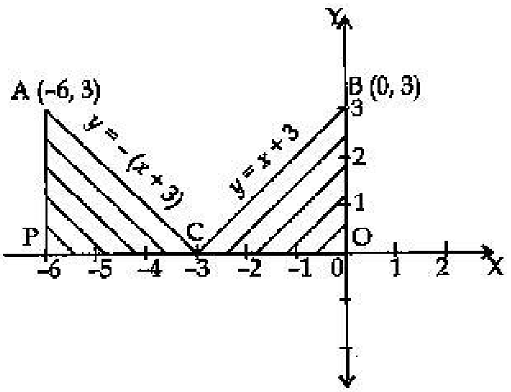
Ans.

26. Sketch the graph of $y = |x+3|$ and evaluate the area under the curve $y = |x+3|$ above x-axis and between $x = -6$ to $x = 0$. **[6]**

Solution : For drawing a sketch of the graph of $y = |x+3|$, we construct the following table of values

x	-6	-5	-3	-2	-1	0
y	3	2	0	1	2	3

Plot these points, a rough sketch is shown in the figure below.



Note that $y = |x+3|$

$$= \begin{cases} -(x+3) & \text{for} \quad x \le -3 \\ x+3 & \text{for} \quad x > 3 \end{cases}$$

So graph consists of two half lines meeting at $x = -3$.

Also $\int_{-6}^{0} |x+3|dx$ area enclosed between graph of $y = |x+3|$, the x-axis and the lines $x = -6$, $x = 0$

$= \text{area } \Delta APC + \text{area } \Delta COB$

$$= \int_{-6}^{-3} (-x-3)dx + \int_{-3}^{0} (x+3)dx$$

$$= \left[\frac{-x^2}{2} - 3x\right]_{-6}^{-3} + \left[\frac{x^2}{2} + 3x\right]_{-3}^{0}$$

$$= \left(-\frac{9}{2} + 9\right) - (-18 + 18) + 0 - \left(\frac{9}{2} - 9\right)$$

$$= \frac{9}{2} + \frac{9}{2} = 9 \text{ sq. units.}$$ **Ans.**

27. Find the distance of the point (–1, –5, –10), from the point of intersection of the line $\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and Plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$. **[6]**

Solution : Equation of the line is

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$$

$$\vec{r} = (2 + 3\lambda)\hat{i} + (-1 + 4\lambda)\hat{j} + (2 + 2\lambda)\hat{k} \quad \text{...(i)}$$

The equation of the given plane is

$$\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5 \quad \text{...(ii)}$$

Since the point of intersection of the line and plane lies on the plane as well as on the line, from (i)

Any point on line is

$$= (2 + 3\lambda, -1 + 4\lambda, 2 + 2\lambda)$$

It lies on the plane (ii)

$$(2 + 3\lambda)(1) + (-1 + 4\lambda)(-1) + (2 + 2\lambda)(1) = 5$$

$$\Rightarrow \quad (2 + 3\lambda + 1 - 4\lambda + 2 + 2\lambda) = 5$$

$$\Rightarrow \quad 5 + \lambda = 5$$

$$\Rightarrow \quad \lambda = 0$$

Substituting the value of λ in (i), we get

$$\vec{r} = [2 + 3(0)]\hat{i} + [-1 + 4(0)]\hat{j} + [2 + 2(0)]\hat{k}$$

$$= 2\hat{i} - \hat{j} + 2\hat{k}$$

Thus, the point of intersection of the given line and plane is (2, –1, 2).

Now, the distance of the point (–1, –5, –10) from the point (2, –1, 2)

$$= \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2}$$

$$= \sqrt{9 + 16 + 144}$$

$$= \sqrt{169} = 13.$$ **Ans.**

28. Given three identical boxes I, II and III each containing two coins. In box I, both coins are gold coins, in box II, both are silver coins and in box III, there is one gold and one silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold ? [6]

Solution : Let E_1 be box I is chosen, E_2 be box II is chosen and E_3 be box III be chosen and A be the coin drawn is of gold.

We have,

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$P(A/E_1)$ = Probability of drawing a gold coin from box I

$$\therefore \quad P(A/E_1) = \frac{2}{2} = 1$$

$P(A/E_2)$ = Probability of drawing a gold coin from box II

$$\therefore \quad P(A/E_2) = 0$$

$P(A/E_3)$ = Probability of drawing a gold coin from box III

$$\therefore \quad P(A/E_3) = \frac{1}{2}$$

$\therefore$ Probability that the other coin in the box is of gold = Probability that gold coin is drawn from the box I

$$= P(E_1/A)$$

$$= \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3} \times 1}{\frac{1}{3} \times 1 + \frac{1}{3} \times 0 + \frac{1}{3} \times \frac{1}{2}} = \frac{2}{3}.$$ **Ans.**

29. A merchant plans to sell two types of personal computer – a desktop model and a portable model that will cost ₹ 25,000 and ₹ 40,000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than ₹ 70 lakhs and his profit on the desktop model is ₹ 4,500 and on the portable model is ₹ 5,000. Make an L.P.P. and solve it graphically. **[6]**

Solution : Let the merchant stock x desktop computers and y portable computers. We construct the following table :

Type	Number	Cost per computer	Investment	Maximum (profit)
Desktop	x	₹25,000	₹25,000 x	₹4,500 x
Portable	y	₹40,000	₹40,000 y	₹5,000 y
	250		₹70,00,000	

∴ The LPP is

Maximize $Z = 4{,}500\,x + 5{,}000\,y$

Subject to constraints :

$$x + y \le 250$$

$$25{,}000\,x + 40{,}000\,y \le 70{,}00{,}000$$

$$\Rightarrow \quad 5x + 8y \le 1{,}400$$

and $\quad x \ge 0\ ,\ y \ge 0$

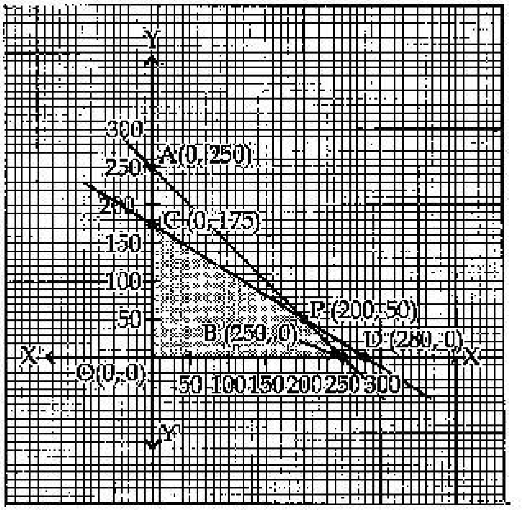
First we draw the lines AB and CD whose equations are

$$x + y = 250 \qquad \text{...(i)}$$

	A	B
x	0	250
y	250	0

and $\quad 5x + 8y = 1400 \qquad$ ...(ii)

	C	D
x	0	280
y	175	0



The feasible region is OBPCO which is shaded in the figure.

The vertices of feasible region are O(0, 0), B(250,0), P(200, 50) and C(0, 175).

P is the point of intersection of the lines.

$$x + y = 250$$

and $\quad 5x + 8y = 1400$

Solving these equation we get point P(200, 50).

∴ The value of objective function

$$Z = 4500\,x + 5000y$$

At these vertices are as follows :

Corner points	Maximize Z = 4500x+5000y
At O(0, 0)	Z = 0
At B(250, 0)	Z = 1125000
At P(200, 50)	Z = 1150000 (maximum)
At C(0, 175)	Z = 875000

Hence, the profit is maximum at ₹11,50,000 when 200 desktop computers and 50 portable computers are stocked. **Ans.**

Mathematics 2011 (Outside Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous set.

SECTION — A

9. Evaluate : $\int \frac{(\log x)^2}{x}dx$ **[1]**

Solution : Let $I = \int \frac{(\log x)^2}{x}dx$

Putting $\log x = t \Rightarrow \frac{1}{x}dx = dt$

$$\therefore \quad I = \int t^2 dt = \frac{t^3}{3} + C$$

$$= \frac{(\log x)^3}{3} + C. \qquad \textbf{Ans.}$$

10. Write a unit vector in the direction of the vector $\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$. **[1]**

Solution : The given vector is

$$\vec{a} = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$|\vec{a}| = \sqrt{(2)^2 + (1)^2 + (2)^2}$$

$$= \sqrt{4+1+4}$$

$$= \sqrt{9} = 3$$

$$\text{Unit vector, } \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3}$$

$$= \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}. \qquad \textbf{Ans.}$$

SECTION – B

19. Prove the following :

$$2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\frac{31}{17}\right) \qquad \textbf{[4]}$$

Solution : L.H.S.

$$= 2\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left[\frac{2.\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}\right] + \tan^{-1}\frac{1}{7}$$

$$\left[\because 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{1}{\frac{3}{4}}\right) + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left[\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}}\right]$$

$$= \tan^{-1}\left(\frac{\frac{31}{21}}{\frac{17}{21}}\right)$$

$$= \tan^{-1}\left(\frac{31}{17}\right) = \text{R.H.S.}$$

Hence Proved.

20. Using properties of determinants, solve the following for x :

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0. \qquad \textbf{[4]}$$

Solution : The determinant is

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} = 0$$

Taking $(3a-x)$ common from C_1

$$(3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$; $R_3 \to R_3 - R_1$, we get

$$(3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 0$$

Expanding along C_1, we get

$$(3a-x)\,[1(2x.2x-0)] = 0$$

$$\Rightarrow \quad 4x^2(3a-x) = 0$$

$$\Rightarrow \quad 4x^2 = 0 \Rightarrow x = 0$$

$$\text{and} \quad 3a - x = 0 \Rightarrow x = 3a$$

Ans.

21. Evaluate : $\int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$. **[4]**

Solution : Let

$$I = \int_0^{\frac{\pi}{4}} \log(1+\tan x)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$\left[\text{using property } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log\left(\frac{2}{1+\tan x}\right)dx$$

$$= \int_0^{\frac{\pi}{4}} \log 2\, dx - \int_0^{\frac{\pi}{4}} \log(1+\tan x)\, dx$$

$$= \log 2\, [x]_0^{\frac{\pi}{4}} - I$$

$$\Rightarrow \quad 2I = \log 2\left[\frac{\pi}{4} - 0\right]$$

$$\Rightarrow \quad I = \frac{\pi}{8}\log 2 \qquad \textbf{Ans.}$$

22. Solve the following differential equation :

$$xdy - (y + 2x^2)dx = 0 \qquad [4]$$

Solution : Given,

$$xdy - (y + 2x^2)dx = 0$$

$$\Rightarrow \quad x\,dy = (y + 2x^2)dx$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y+2x^2}{x}$$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{1}{x}y = 2x \qquad ...(i)$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

Here, $P = -\frac{1}{x}$ and $Q = 2x$

Integrating factor,

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int\left(-\frac{1}{x}\right)} = e^{-\log x}$$

$$= e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$\therefore$ The solution is given by

$$y \times \text{I. F.} = \int (Q \times \text{I.F.})\,dx + C$$

$$\Rightarrow \quad \frac{y}{x} = \int 2x.\frac{1}{x}dx + C$$

$$\Rightarrow \quad y.\frac{1}{x} = 2x + C$$

$$\Rightarrow \quad y = 2x^2 + Cx. \qquad \textbf{Ans.}$$

SECTION – C

28. Using matrices, solve the following system of equations : **[6]**

$$x + 2y + z = 7$$
$$x + 3z = 11$$
$$2x - 3y = 1$$

Solution : The given equations are

$$x + 2y + z = 7$$
$$x + 3z = 11$$
$$2x - 3y = 1$$

The given system of equations can be written in matrix form

$$AX = B \qquad ...(i)$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

where, $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ 2 & -3 & 0 \end{vmatrix}$

$= 1(0+9) - 2(0-6) + 1(-3-0) = 21 - 3 = 18 \neq 0$

$\therefore$ A^{-1} exists

Cofactors of A,

$A_{11} = 0 + 9 = 9$, $A_{12} = -(0-6) = 6$, $A_{21} = -(0+3) = -3$
$A_{13} = -3 - 0 = -3$, $A_{22} = 0 - 2 = -2$,
$A_{23} = -(-3-4) = 7$, $A_{31} = 6 - 0 = 6$,
$A_{32} = -(3-1) = -2$, $A_{33} = 0 - 2 = -2$

$$\text{adj } A = \begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$= \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}$$

$$\therefore \quad X = A^{-1}B \qquad [\text{ from (i)}]$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 9 & -3 & 6 \\ 6 & -2 & -2 \\ -3 & 7 & -2 \end{bmatrix}\begin{bmatrix} 7 \\ 11 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 36 \\ 18 \\ 54 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\Rightarrow \quad x = 2, y = 1, z = 3. \qquad \textbf{Ans.}$$

29. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = 1$ and $\vec{r}.(2\hat{i}+3\hat{j}-\hat{k}) + 4 = 0$ and parallel to x-axis. **[6]**

Solution : The given equation of planes are

$$\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = 1$$

and $\vec{r}.(2\hat{i}+3\hat{j}-\hat{k})+4=0$

The cartesian equation of the planes are

$$x+y+z-1 = 0 \quad ...(i)$$
$$2x+3y-z+4 = 0 \quad ...(ii)$$

Equation of plane passing through the intersection of the plane (i) and (ii) is

$$(x+y+z-1)+\lambda(2x+3y-z+4)=0 \quad ...(iii)$$

$\Rightarrow \quad x+y+z-1+2\lambda x+3\lambda y-\lambda z+4\lambda=0$

$\Rightarrow \quad x+2\lambda x+y+3\lambda y+z-\lambda z-1+4\lambda=0$

$\Rightarrow \quad (1+2\lambda)x+(1+3\lambda)y+(1-\lambda)z-1+4\lambda=0$

$\therefore$ Dr's of the normal to the plane are $1+2\lambda$, $1+3\lambda$, $1-\lambda$

This plane is parallel to x–axis

$\therefore \quad (1+2\lambda)(1)+(1+3\lambda)(0)+(1-\lambda)(0)=0$

[$\because$ $d.r'.s$ of x–axis are 1, 0, 0]

$\Rightarrow \quad 1+2\lambda=0$

$\Rightarrow \quad \lambda=-\frac{1}{2}$

Putting the value of λ in (iii), we get

$$(x+y+z-1)-\frac{1}{2}(2x+3y-z+4)=0$$

$\Rightarrow \quad 2x+2y+2z-2-2x-3y+z-4=0$

$\Rightarrow \quad -y+3z-6=0$

$\therefore \quad y-3z+6=0.$

Ans.

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Mathematics 2011 (Outside Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

Note : Except for the following questions, all the remaining questions have been asked in previous sets.

SECTION – A

1. Evaluate : $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx.$ **[1]**

Solution : Let $I = \int \frac{e^{\tan^{-1}x}}{1+x^2}dx$

Putting $\tan^{-1}x=t$

$\Rightarrow \quad \frac{1}{1+x^2}.dx = dt$

$\therefore \quad I = \int e^t\,dt = e^t+C$

$= e^{\tan^{-1}x}+C.$ **Ans.**

2. Write the angle between two vectors $\vec{a}$ and $\vec{b}$ magnitudes with $\sqrt{3}$ and 2 respectively having $\vec{a}.\vec{b} = \sqrt{6}$. **[1]**

Solution : Given,

$$\vec{a}.\vec{b} = \sqrt{6}, |\vec{a}|=\sqrt{3}, |\vec{b}|=2$$

Angle between $\vec{a}$ and $\vec{b}$ is

$$\cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\frac{\sqrt{6}}{\sqrt{3}\times 2}\times\frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{18}}{3\times 2}=\frac{3\sqrt{2}}{3\times 2}$$

$\Rightarrow \quad \cos\theta = \frac{1}{\sqrt{2}}$

$\therefore \quad \theta = \cos^{-1}\frac{1}{\sqrt{2}} \Rightarrow \theta=\frac{\pi}{4}.$ **Ans.**

SECTION – B

11. Prove that : $\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)=\frac{\pi}{4}$ **[4]**

Solution : L.H.S.

$$= \tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}.\frac{1}{5}}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$\left[\because \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{\frac{7}{10}}{\frac{9}{10}}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{7}{9}\right)+\tan^{-1}\left(\frac{1}{8}\right)$$

$$= \tan^{-1}\left(\frac{\frac{7}{9}+\frac{1}{8}}{1-\frac{7}{9}.\frac{1}{8}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{65}{72}}{\frac{65}{72}}\right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4} = \text{R.H.S.}$$

Hence Proved.

12. Using properties of determinants, solve the following for x:

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0. \qquad [4]$$

Solution :

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 3x+a & x & x \\ 3x+a & x+a & x \\ 3x+a & x & x+a \end{vmatrix} = 0$$

Taking $(3x + a)$ common from C_1, we get

$$(3x+a)\begin{vmatrix} 1 & x & x \\ 1 & x+a & x \\ 1 & x & x+a \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1$; and $R_3 \to R_3 - R_1$, we get

$$(3x+a)\begin{vmatrix} 1 & x & x \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = 0$$

Expanding along C_1, we get

$$(3x + a)\,[1(a^2 - 0)] = 0$$

$$\Rightarrow \quad a^2(3x + a) = 0$$

$$\Rightarrow \quad 3x + a = 0$$

$$\Rightarrow \quad x = -\frac{a}{3}. \qquad \textbf{Ans.}$$

13. Evaluate : $\int_0^1 \log\left(\frac{1}{x}-1\right)dx.$ **[4]**

Solution : Let $I = \int_0^1 \log\left(\frac{1-x}{x}\right)dx \qquad ...(i)$

$$= \int_0^1 \log\left(\frac{1-(1-x)}{1-x}\right).dx$$

$$\left[\because \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$= \int_0^1 \log\left(\frac{x}{1-x}\right).dx \qquad ...(ii)$$

Adding (i) and (ii), we get

$$2I = \int_0^1 \log\left(\frac{x}{1-x}.\frac{1-x}{x}\right).dx$$

$$= \int_0^1 \log 1.dx$$

$$= \int_0^1 0\,dx = 0$$

$$\therefore \quad I = 0 \qquad \textbf{Ans.}$$

14. Solve the following differential equation :
$xdy + (y - x^3)dx = 0$ **[4]**

Solution : We have,

$$x.dy + (y - x^3).dx = 0$$

or $$x.\frac{dy}{dx} + (y - x^3) = 0$$

or $$\frac{dy}{dx} + \frac{y}{x} = x^2$$

This is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q,$$

Here, $P = \frac{1}{x}$ and $Q = x^2$

$$\text{I.F.} = e^{\int P dx}$$

$$= e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

$\therefore$ The solution is given by

$$y \times \text{I.F.} = \int Q.(\text{I.F.}).dx + C$$

$$yx = \int x^2.x.dx + C$$

$$\Rightarrow \quad yx = \int x^3.dx + C$$

$$\Rightarrow \quad yx = \frac{x^4}{4} + C$$

$$\Rightarrow \quad y = \frac{x^3}{4} + \frac{C}{x} \qquad \textbf{Ans.}$$

SECTION – C

23. Using matrices, solve the following system of equations : **[6]**

$$x + 2y - 3z = -4$$
$$2x + 3y + 2z = 2$$
$$3x - 3y - 4z = 11$$

Solution : The given system of equations can be written in matrix form as

$$AX = B \quad \text{...(i)}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{vmatrix}$

$$= 1(-12+6) - 2(-8-6) - 3(-6-9)$$
$$= -6 + 28 + 45 = 67 \neq 0$$

$\therefore A^{-1}$ exists.

For adj A, cofactors are

$A_{11} = -12 + 6 = -6$, $A_{12} = -(-8-6) = 14$,
$A_{13} = (-6-9) = -15$, $A_{21} = -(-8-9) = 17$,
$A_{22} = -4 + 9 = 5$, $A_{23} = -(-3-6) = 9$
$A_{31} = 4 + 9 = 13$, $A_{32} = -(2+6) = -8$,
$A_{33} = 3 - 4 = -1$

$$\therefore \quad \text{adj } A = \begin{bmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{adj } A$$

$$= \begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}$$

$\therefore \quad X = A^{-1}B$ [from (i)]

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{67}\begin{bmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \\ 11 \end{bmatrix}$$

$$= \frac{1}{67}\begin{bmatrix} 201 \\ -134 \\ 67 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

$\Rightarrow \quad x = 3, y = -2$ and $z = 1$. **Ans.**

24. Find the equation of the plane passing through the line of intersection of the planes $2x + y - z = 3$ and $5x - 3y + 4z + 9 = 0$ and parallel to the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}$. **[6]**

Solution : Given planes are

$$2x + y - z - 3 = 0 \quad \text{...(i)}$$
$$5x - 3y + 4z + 9 = 0 \quad \text{..(ii)}$$

Any plane passing through the line of intersection of (i) and (ii) can be taken as

$$2x + y - z - 3 + \lambda(5x - 3y + 4z + 9) = 0$$

$$(2+5\lambda)x + (1-3\lambda)y + (-1+4\lambda)z - 3 + 9\lambda = 0 \text{...(iii)}$$

This plane is parallel to the line

$$\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-5}{5}.$$

If $\quad 2(2+5\lambda) + 4(1-3\lambda) + 5(-1+4\lambda) = 0$

$\Rightarrow \quad 4 + 10\lambda + 4 - 12\lambda - 5 + 20\lambda = 0$

$\Rightarrow \quad 18\lambda + 3 = 0$

$\Rightarrow \quad \lambda = -\frac{1}{6}$

Substituting this value of λ in (iii), we get the required plane as

$$\left(2 - \frac{5}{6}\right)x + \left(1 + \frac{3}{6}\right)y + \left(-1 - \frac{4}{6}\right)z - 3 - \frac{9}{6} = 0$$

$\Rightarrow \quad \frac{7}{6}x + \frac{9}{6}y - \frac{10}{6}z - \frac{27}{6} = 0$

$\Rightarrow \quad 7x + 9y - 10z - 27 = 0.$

Ans.

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Mathematics 2011 (Delhi) SET I

Time allowed : 3 hours Maximum marks : 100

SECTION — A

1. State the reason for the relation R in the set {1, 2, 3} given by R = {(1, 2), (2, 1)} not to be transitive. [1]

Solution : In the case of transitive relation

If (a, b) and $(b, c) \in R$

$\Rightarrow \quad (a, c) \in R$

Here, (1, 2) and (2, 1) $\in$ R but (1, 1) $\notin$ R.

So, R is not transitive. **Ans.**

2. Write the value of $\sin\left(\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$. [1]

Solution : We have, $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$

$$= \sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\sin\frac{\pi}{6}\right)\right]$$

$$= \sin\left[\frac{\pi}{3} - \sin^{-1}\left\{\sin\left(-\frac{\pi}{6}\right)\right\}\right]$$

$$= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin\frac{3\pi}{6} = \sin\frac{\pi}{2}$$

$$= 1.$$ **Ans.**

3. For a 2 × 2 matrix, A = $[a_{ij}]$, whose elements are given by $a_{ij} = \frac{i}{j}$, write the value of a_{12}. [1]

Solution : The order of the given matrix is 2 × 2. So,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$$

$$a_{ij} = \frac{i}{j}$$

Put $i = 1$ and $j = 2$

$\therefore \quad a_{12} = \frac{1}{2}$. **Ans.**

4. For what value of x, the matrix $\begin{bmatrix} 5-x & x+1 \\ 2 & 4 \end{bmatrix}$ is singular? [1]

Solution : Matrix A is singular if $|A| = 0$

$$\Rightarrow \begin{vmatrix} 5-x & x+1 \\ 2 & 4 \end{vmatrix} = 0$$

$\Rightarrow \quad 4(5-x) - 2(x+1) = 0$

$\Rightarrow \quad 20 - 4x - 2x - 2 = 0$

$\Rightarrow \quad x = 3.$ **Ans.**

5. Write A^{-1} for $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$. [1]

Solution : We know that,

$$A^{-1} = \frac{1}{|A|} \times \text{adj } A$$

$$\text{adj } A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

and $\quad |A| = 6 - 5 = 1$

$$A^{-1} = \frac{1}{1} \times \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}.$$ **Ans.**

6. Write the value of $\int \sec x(\sec x + \tan x)dx$. [1]

Solution : Let $\quad I = \int \sec x(\sec x + \tan x)dx.$

$$= \int \sec^2 x\,dx + \int \sec x.\tan x\,dx$$

$$= \tan x + \sec x + C.$$ **Ans.**

7. Write the value of $\int \frac{dx}{x^2+16}$ [1]

Solution : Let $\quad I = \int \frac{dx}{x^2 + (4)^2}$

$$= \frac{1}{4}\tan^{-1}\frac{x}{4} + C.$$ **Ans.**

8. For what value of 'a' the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$ and $a\hat{i} + 6\hat{j} - 8\hat{k}$ are collinear? [1]

Solution : Let,

$\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$ and $\vec{b} = a\hat{i} + 6\hat{j} - 8\hat{k}$

For collinear vectors

$$\vec{b} = \lambda\vec{a}, \text{ here } \lambda = -2$$

$$a\hat{i} + 6\hat{j} - 8\hat{k} = \lambda(2\hat{i} - 3\hat{j} + 4\hat{k})$$

On comparing, we get

$6 = -3\lambda$

$\Rightarrow \quad \lambda = -2$

Also $\quad a = 2\lambda$

$\Rightarrow \quad a = -4$ **Ans.**

9. Write the direction cosines of the vector $-2\hat{i} + \hat{j} - 5\hat{k}$. [1]

Solution : Direction cosines of the vector $-2\hat{i} + \hat{j} - 5\hat{k}$ are

$$\frac{-2}{\sqrt{(-2)^2+1^2+(-5)^2}}, \frac{1}{\sqrt{(-2)^2+1^2+(-5)^2}}, \frac{-5}{\sqrt{(-2)^2+1^2+(-5)^2}}$$

i.e., $\frac{-2}{\sqrt{30}}, \frac{1}{\sqrt{30}}, \frac{-5}{\sqrt{30}}$. **Ans.**

10. Write the intercept cut off by the plane $2x+y-z=5$ on x-axis. **[1]**

Solution : Given equation of plane is

$$2x+y-z = 5 \quad ...(i)$$

Intercept on x-axis *i.e.,* $y=0, z=0$

$$2x+0-0 = 5 \quad \text{[from (i)]}$$

$$\therefore \quad x = \frac{5}{2}.$$ **Ans.**

SECTION – B

11. Consider the binary operation * on the set {1, 2, 3, 4, 5} defined by $a * b = \min. \{a, b\}$. Write the operation table of the operation *.** **[4]**

12. Prove the following :

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2}, x \in \left(0, \frac{\pi}{4}\right)$$

[4]

Solution : L.H.S.

$$= \cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right); \quad x \in \left(0, \frac{\pi}{4}\right)$$

$$= \cot^{-1}\left(\frac{(\sqrt{1+\sin x}+\sqrt{1-\sin x})(\sqrt{1+\sin x}+\sqrt{1-\sin x})}{(\sqrt{1+\sin x}-\sqrt{1-\sin x})(\sqrt{1+\sin x}+\sqrt{1-\sin x})}\right)$$

$$= \cot^{-1}\left(\frac{1+\sin x+1-\sin x+2\sqrt{(1+\sin x)(1-\sin x)}}{(1+\sin x)-(1-\sin x)}\right)$$

$$= \cot^{-1}\left(\frac{2+2\sqrt{1-\sin^2 x}}{1+\sin x-1+\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2(1+\cos x)}{2\sin x}\right)$$

$$= \cot^{-1}\left(\frac{2\cos^2 x/2}{2\sin x/2 . \cos x/2}\right)$$

$$\left[\because 1+\cos x = 2\cos^2 x/2\right]$$

$$= \cot^{-1}\left(\frac{\cos x/2}{\sin x/2}\right)$$

$$= \cot^{-1}(\cot x/2) = x/2 = \text{R.H.S.}$$

Hence Proved.

OR

Find the value of $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$.

Solution : Given,

$$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$$

$$= \tan^{-1}\left[\frac{\frac{x}{y}-\left(\frac{x-y}{x+y}\right)}{1+\frac{x}{y}\left(\frac{x-y}{x+y}\right)}\right]$$

$$\left[\because \tan^{-1}x-\tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left[\frac{(x^2+xy-xy+y^2)/(x+y)y}{(xy+y^2+x^2-xy)/(x+y)y}\right]$$

$$= \tan^{-1}\left[\frac{x^2+y^2}{x^2+y^2}\right] = \tan^{-1}(1)$$

$$= \tan^{-1}\left(\tan\frac{\pi}{4}\right) = \frac{\pi}{4}.$$ **Ans.**

13. Using properties of determinants, prove that

$$\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix} = 4a^2b^2c^2.$$ **[4]**

Solution : L.H.S.

$$= \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

Taking a, b, c common from R_1, R_2 and R_3 respectively.

$$\Delta = abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix}$$

Taking a, b, c common from C_1, C_2 and C_3 respectively.

$$\Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$

$$= a^2b^2c^2\begin{vmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

**Answer is not given due to the change in present syllabus

Expanding along C_1, we get

$a^2b^2c^2(2\times 2) = 4a^2b^2c^2$ = R.H.S.

Hence Proved.

14. Find the value of 'a' for which the function f defined as

$$f(x) = \begin{cases} a\sin\frac{\pi}{2}(x+1), & x\le 0 \\ \frac{\tan x-\sin x}{x^3}, & x>0 \end{cases}$$

is continuous at $x=0$. **[4]**

Solution : L.H.L $= \lim_{x\to 0^-}\left\{a\sin\frac{\pi}{2}(x+1)\right\}$

$$= a\sin\frac{\pi}{2} = a\times 1 = a$$

$$\text{R.H.L} = \lim_{x\to 0^+}\left(\frac{\tan x-\sin x}{x^3}\right)$$

$$= \lim_{x\to 0^+}\frac{\sin x(1-\cos x)}{x^3.\cos x}$$

$$= \lim_{x\to 0^+}\frac{\sin x}{x}.\lim_{x\to 0^+}\frac{2\sin^2\frac{x}{2}}{x^2}.\lim_{x\to 0^+}\frac{1}{\cos x}$$

$$\left(\because \cos x = 1-2\sin^2\frac{x}{2}\right)$$

$$= \lim_{x\to 0^+}\frac{\sin x}{x}.\frac{1}{2}\lim_{x\to 0^+}\left[\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right]^2.\lim_{x\to 0^+}\frac{1}{\cos x}$$

$$= 1.\frac{1}{2}.1.1 = \frac{1}{2}$$

Since, $f(x)$ is continuous

$\therefore$ L.H.L = R.H.L.

$$a = \frac{1}{2}$$ **Ans.**

15. Differentiate $x^{x\cos x} + \frac{x^2+1}{x^2-1}$ w.r.t. x. **[4]**

Solution : Given, $x^{x\cos x} + \frac{x^2+1}{x^2-1}$

Let $y = u + v$

$$\Rightarrow \frac{dy}{dx} = \frac{du}{dx}+\frac{dv}{dx} \quad ...(i)$$

Here, $u = x^{x\cos x}$

Taking log of both sides, we get

$$\log u = x\cos x\log x$$

Differentiating w.r.t. x, we get

$$\frac{1}{u}\frac{du}{dx} = x\cos x\frac{d}{dx}(\log x) + \log x\frac{d}{dx}(x\cos x)$$

$$\Rightarrow \frac{d}{dx} = u\left[x\cos x\times\frac{1}{x}+\log x[x(-\sin x)+\cos x\times 1]\right]$$

$$= x^{x\cos x}[\cos x - x\log x\sin x + \log x\cos x] \quad ...(ii)$$

and $v = \frac{x^2+1}{x^2-1}$

Differentiating w.r.t. x, we get

$$\frac{d}{dx} = \frac{(x^2-1)\frac{d}{dx}(x^2+1)-(x^2+1)\frac{d}{dx}(x^2-1)}{(x^2-1)^2}$$

$$= \frac{(x^2-1)\times 2x-(x^2+1)\times 2x}{(x^2-1)^2}$$

$$= \frac{2x(x^2-1-x^2-1)}{(x^2-1)^2}$$

$$\Rightarrow \frac{d}{dx} = \frac{-4x}{(x^2-1)^2} \quad ...(iii)$$

From (i), (ii) and (iii),

$$\frac{dy}{dx} = x^{x\cos x}[\cos x - x\log x\sin x + \log x\cos x] - \left(\frac{4x}{(x^2-1)^2}\right).$$ **Ans.**

OR

If $x = a(\theta-\sin\theta)$, $y = a(1+\cos\theta)$ find $\frac{d^2y}{dx^2}$.

Solution : Given,

$$x = a(\theta-\sin\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{dx}{d\theta} = a(1-\cos\theta) \quad ...(i)$$

and $y = a(1+\cos\theta)$

Differentiating w.r.t. θ, we get

$$\frac{dy}{d\theta} = a(-\sin\theta) \quad ...(ii)$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a(-\sin\theta)}{a(1-\cos\theta)}$$

$$= \frac{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

$$\therefore \quad \frac{dy}{dx} = -\cot\frac{\theta}{2}$$

Differentiating w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}.\frac{d\theta}{dx}$$

$$= \frac{1}{2\sin^2\frac{\theta}{2}}.\frac{1}{a(1-\cos\theta)}$$

$$= \frac{1}{2\sin^2\frac{\theta}{2}}.\frac{1}{2a\sin^2\frac{\theta}{2}}$$

$$= \frac{1}{4a}\operatorname{cosec}^4\frac{\theta}{2}.$$ **Ans.**

16. Sand is pouring from a pipe at the rate of 12 cm³/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm? [4]

Solution : Let r be the radius, h be the height and V be the volume of the sand cone.

$$\therefore \quad \frac{dh}{dt} = 12\text{ cm}^3/\text{s and } h = \frac{1}{6}r$$

$$h = 4\text{ cm}$$

Volume of sand cone

$$V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi(6h)^2 h\right)$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi\times 36\times h^3\right)$$

$$\therefore \quad \frac{dh}{dt} = \frac{3}{3}\times\pi h^2\times 36\times\frac{dh}{dt}$$

$$\Rightarrow \quad 12 = \pi(4)^2\times 36\times\frac{dh}{dt}$$

$$\left[\because \frac{dV}{dt} = 12\text{ cm}^3/\text{s}, h = 4\text{ cm}\right]$$

$$\Rightarrow \quad \frac{dh}{dt} = \frac{1}{48\pi}\text{cm/s.}$$ **Ans.**

OR

Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to x–axis.

Solution : When the tangent is parallel to x–axis

$$\frac{dy}{dx} = 0$$

We have

$$x^2 + y^2 - 2x - 3 = 0 \quad \text{...(i)}$$

Differentiating w.r.t. x, we get

$$2x + 2y\frac{dy}{dx} - 2 - 0 = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2 - 2x$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = 2(1-x)$$

$$\Rightarrow \quad y\times 0 = (1-x) \quad \left[\because \frac{dy}{dx} = 0\right]$$

$$\therefore \quad x = 1$$

Putting, $x = 1$ in equation (i), we get $y = \pm 2$

$\therefore$ The required points are $(1, 2)$ and $(1, -2)$.

Ans.

17. Evaluate : $\int \frac{5x+3}{\sqrt{x^2+4x+10}}dx.$ [4]

Solution : Let, $I = \int \frac{5x+3}{\sqrt{x^2+4x+10}}dx$

Let $\quad 5x + 3 = A\frac{d}{dx}(x^2+4x+10) + B$

$$\Rightarrow \quad 5x + 3 = A(2x+4) + B$$

$$\Rightarrow \quad 5x + 3 = 2Ax + (4A + B)$$

Comparing the coefficient of x and constant on both sides, we get

$$2A = 5 \Rightarrow A = \frac{5}{2}$$

and $\quad 4A + B = 3$

$$\Rightarrow \quad 4\times\frac{5}{2} + B = 3$$

$$\Rightarrow \quad B = -7$$

$$I = \frac{5}{2}\int\frac{2x+4}{\sqrt{x^2+4x+10}}dx$$

$$-7\int\frac{1}{\sqrt{x^2+4x+10}}dx \quad \text{...(i)}$$

Putting $x^2 + 4x + 10 = t$ in first term,

$$\Rightarrow \quad (2x+4)dx = dt$$

$$\therefore \quad I = \frac{5}{2}\int\frac{1}{\sqrt{t}}dt - 7\int\frac{1}{\sqrt{x^2+4x+10}}dx$$

$$= \frac{5}{2}\times 2\sqrt{t} - 7\frac{1}{\sqrt{(x+2)^2+(\sqrt{6})^2}}$$

$$= 5\sqrt{x^2+4x+10}$$

$$-7\log\left|(x+2)+\sqrt{(x+2)^2+(\sqrt{6})^2}\right|+C$$

$$= 5\sqrt{x^2+4x+10}$$

$$-7\log\left|(x+2)+\sqrt{x^2+4x+10}\right|+C.$$

Ans.

OR

Evaluate : $\int\frac{2x}{\sqrt{(x^2+1)(x^2+3)}}dx.$

Solution : Let, I

$$= \int\frac{2x}{\sqrt{(x^2+1)(x^2+3)}}dx$$

Putting $x^2 = t$

$\Rightarrow$ $2x\,dx = dt$

$\therefore$ $I = \int\frac{1}{(1+t)(3+t)}dt$

Let $\frac{1}{(1+t)(3+t)} = \frac{A}{1+t}+\frac{B}{3+t}$

$$1 = A(3+t)+B(1+t)$$

Putting $t=-3$

$\Rightarrow$ $B=-\frac{1}{2}$

and putting $t = -1$

$\Rightarrow$ $A=\frac{1}{2}$

$\therefore$ $\frac{1}{(1+t)(3+t)} = \frac{1/2}{1+t}-\frac{1/2}{3+t}$

$$I = \frac{1}{2}\int\frac{dt}{1+t}-\frac{1}{2}\int\frac{1}{3+t}dt$$

$$= \frac{1}{2}\log|1+t|-\frac{1}{2}\log|3+t|+C$$

$$=\frac{1}{2}\log|1+x^2|-\frac{1}{2}\log|3+x^2|+C$$

$$=\frac{1}{2}\log\left|\frac{1+x^2}{3+x^2}\right|+C.$$

Ans.

18. Solve the following differential equation :

$$e^x\tan y\,dx + (1-e^x)\sec^2 y.dy = 0 \qquad [4]$$

Solution : Given, $e^x\tan y\,dx+(1-e^x)\sec^2 y.dy = 0$

$\Rightarrow$ $e^x\tan y\,dx = -(1-e^x)\sec^2 y.dy$

$\Rightarrow$ $\frac{e^x}{1-e^x}dx = -\frac{\sec^2 y}{\tan y}dy$

Integrating both sides, we get

$$\int\frac{e^x}{1-e^x}dx = -\int\frac{\sec^2 y}{\tan y}dy$$

Putting $1-e^x = t \Rightarrow e^x.dx = -dt$

and $\tan y = z \Rightarrow \sec^2 y.dy = dz$

$\Rightarrow$ $-\int\frac{dt}{t} = -\int\frac{dz}{z}$

$\Rightarrow$ $\log|t| = \log|z|+C$

$\therefore$ $\log|1-e^x| = \log|\tan y|+C.$ **Ans.**

19. Solve the following differential equation :

$$\cos^2 x\frac{dy}{dx}+y = \tan x. \qquad [4]$$

Solution : Given,

$$\cos^2 x\frac{dy}{dx}+y = \tan x$$

Dividing by $\cos^2 x$ on both sides, we get

$$\frac{dy}{dx}+\frac{y}{\cos^2 x} = \frac{\tan x}{\cos^2 x}$$

$$\frac{dy}{dx}+\sec^2 x.y = \tan x\sec^2 x \qquad ...(i)$$

This is a linear differential equation of the form

$$\frac{dy}{dx}+Py = Q$$

where $P = \sec^2 x$ and $Q = \tan x\sec^2 x$

$$\text{I.F.} = e^{\int P dx} = e^{\int\sec^2 x dx} = e^{\tan x}$$

$\therefore$ Solution is given by

$$y\times\text{I.F.} = \int\text{I.F.}\times Q.\,dx$$

$$y.e^{\tan x} = \int e^{\tan x}\times\tan x\sec^2 x\,dx$$

Putting $\tan x = t$

$\Rightarrow$ $\sec^2 x.\,dx = dt$

$\Rightarrow$ $y\times e^{\tan x} = \int e^t\times t.dt$

$$= t\int e^t dt-\int\left[\frac{d}{dt}(t)\int e^t.dt\right]dt$$

$\Rightarrow$ $y\,e^{\tan x} = t\times e^t-e^t+C$

$\Rightarrow$ $y\,e^{\tan x} = \tan x\,e^{\tan x}-e^{\tan x}+C$

$\Rightarrow$ $y.e^{\tan x} = e^{\tan x}[\tan x-1]+C$

$\therefore$ $y = (\tan x-1)+Ce^{-\tan x}.$

Ans.

20. Find a unit vector perpendicular to each of the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, where $\vec{a} = 3\hat{i}+2\hat{j}+2\hat{k}$

and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$. **[4]**

Solution : Given,

$$\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$$

and $$\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\therefore \quad \vec{a} + \vec{b} = 3\hat{i} + 2\hat{j} + 2\hat{k} + \hat{i} + 2\hat{j} - 2\hat{k}$$

$$= 4\hat{i} + 4\hat{j} + 0\hat{k}$$

and $$\vec{a} - \vec{b} = 3\hat{i} + 2\hat{j} + 2\hat{k} - \hat{i} - 2\hat{j} + 2\hat{k}$$

$$= 2\hat{i} + 0\hat{j} + 4\hat{k}$$

Now,

$$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$= 16\hat{i} - 16\hat{j} + (-8)\hat{k}$$

$$= 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$$\therefore |(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})| = \sqrt{(16)^2 + (-16)^2 + (-8)^2}$$

$$= \sqrt{256 + 256 + 64}$$

$$= \sqrt{576} = 24$$

$\therefore$ Required perpendicular unit vector

$$= \frac{(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})}{|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})|}$$

$$= \frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24}$$

$$= \frac{16}{24}\hat{i} - \frac{16}{24}\hat{j} - \frac{8\hat{k}}{24} = \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}.$$ **Ans.**

21. Find the angle between the following pair of lines :

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$$

and $$\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$$

and check whether the lines are parallel or perpendicular. **[4]**

Solution : Given lines are $\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3}$

and $$\frac{x+2}{-1} = \frac{2y-8}{4} = \frac{z-5}{4}$$

Writing equation in standard form

$$\frac{-x+2}{-2} = \frac{y-1}{7} = \frac{z+3}{-3} \qquad ...(i)$$

and $$\frac{x+2}{-1} = \frac{y-2}{2} = \frac{z-5}{4} \qquad ...(ii)$$

Here, $$\vec{b_1} = 2\hat{i} + 7\hat{j} - 3\hat{k}$$

$$\vec{b_2} = -\hat{i} + 2\hat{j} + 4\hat{k}$$

Angle between lines is

$$\cos\theta = \frac{\vec{b}_1 . \vec{b}_2}{|\vec{b}_1||\vec{b}_2|}$$

$$\vec{b_1} \cdot \vec{b_2} = (2\hat{i} + 7\hat{j} - 3\hat{k}).(-\hat{i} + 2\hat{j} + 4\hat{k})$$

$$= -2 + 14 - 12 = 0$$

$$|\vec{b}_1| = \sqrt{4+49+9} = \sqrt{62}$$

$$|\vec{b}_2| = \sqrt{1+4+16} = \sqrt{21}$$

$$\cos\theta = \frac{0}{\sqrt{62}.\sqrt{21}} = 0$$

$$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{2}$$

$$\therefore \quad \theta = \frac{\pi}{2}.$$

For perpendicularity,

$$a_1 = 2, b_1 = 7, c_1 = -3$$

$$a_2 = -1, b_2 = 2, c_2 = 4$$

We know that

$$a_1 \times a_2 + b_1 \times b_2 + c_1 \times c_2$$

$$= 2 \times (-1) + 7 \times 2 - 3 \times 4$$

$$= -14 + 14 = 0$$

Hence, the lines are perpendicular. **Ans.**

22. Probabilities of solving a specific problem independently by A and B are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that (i) the problem is solved (ii) exactly one of them solves the problem. **[4]**

Solution : Given,

$$P(A) = \frac{1}{2}$$

and $$P(B) = \frac{1}{3}$$

$$\therefore \quad P(\bar{A}) = 1 - P(A)$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(\bar{B}) = 1 - P(B)$$

$$= 1-\frac{1}{3}=\frac{2}{3}$$

(i) P (the problem is solved)

$$= P(\text{atleast one of them will solve})$$
$$= P(A\cup B)$$
$$= 1-P(\overline{A\cup B})$$
$$= 1-P(\overline{A}\cap\overline{B})$$
$$= 1-P(\overline{A})P(\overline{B})$$
$$= 1-\frac{1}{2}\times\frac{2}{3}=\frac{2}{3}.$$

(ii) P (exactly one of them solved)

$$= P(A)P(\overline{B})+P(\overline{A})P(B)$$
$$=\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{1}{3}=\frac{1}{2}\left(\frac{2}{3}+\frac{1}{3}\right)=\frac{1}{2}\times 1=\frac{1}{2}.$$

Ans.

SECTION – C

23. Using matrix method, solve the following system of equations :

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z} = 4, \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1,$$
$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z} = 2; x, y, z\neq 0.$$ **[6]**

Solution : The given system of equations are

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z} = 4,\ \frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1,$$
$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z} = 2$$

Given equations can be written as $AX = B$

$$\begin{bmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{bmatrix}\begin{bmatrix}1/x\\ 1/y\\ 1/z\end{bmatrix}=\begin{bmatrix}4\\ 1\\ 2\end{bmatrix},$$

where $A=\begin{bmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{bmatrix}, X=\begin{bmatrix}1/x\\ 1/y\\ 1/z\end{bmatrix}$ and $B=\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$

$$|A| = [2+(120-45)-3(-80-30)+10(36-36)]$$
$$= [150+330+720] = 1200\neq 0$$

$\therefore A^{-1}$ exists.

Cofactors of A,

$A_{11} = (120-45) = 75,\quad A_{13} = (36+36) = 72,$

$A_{12} = -(-80-30) = 110,$

$A_{21} = -(-60-90) = 150,\ A_{23} = -(18-18) = 0,$

$A_{22} = (-40-60) = -100,$

$A_{31} = (15+60) = 75,\quad A_{33} = (-12-12) = -24,$

$A_{32} = -(10-40) = 30.$

$$\therefore\quad \text{adj } A = \begin{bmatrix}75 & 110 & 72\\ 150 & -100 & 0\\ 75 & 30 & -24\end{bmatrix}^T$$
$$= \begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$$
$$A^{-1} = \frac{1}{|A|}\times \text{adj } A$$
$$\Rightarrow\quad A^{-1} = \frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$$
$$\therefore\quad X = A^{-1}B$$
$$\Rightarrow\quad \begin{bmatrix}1/x\\ 1/y\\ 1/z\end{bmatrix} = \frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$$
$$= \frac{1}{1200}\begin{bmatrix}600\\ 400\\ 240\end{bmatrix}=\begin{bmatrix}1/2\\ 1/3\\ 1/5\end{bmatrix}$$
$$\frac{1}{x}=\frac{1}{2}\Rightarrow x=2$$
$$\frac{1}{y}=\frac{1}{3}\Rightarrow y=3$$
and $$\frac{1}{z}=\frac{1}{5}\Rightarrow z=5.$$ Ans.

OR

Using elementary transformations, find the inverse of the matrix $\begin{bmatrix}1 & 3 & -2\\ -3 & 0 & -1\\ 2 & 1 & 0\end{bmatrix}$

Solution : Given

$$A = \begin{bmatrix}1 & 3 & -2\\ -3 & 0 & -1\\ 2 & 1 & 0\end{bmatrix}$$

We have $A = IA$

$$\begin{bmatrix}1 & 3 & -2\\ -3 & 0 & -1\\ 2 & 1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A$$

Applying $R_2\to R_2+3R_1$ and $R_3\to R_3-2R_1$, we get

$$\begin{bmatrix}1 & 3 & -2\\ 0 & 9 & -7\\ 0 & -5 & 4\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 3 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}A$$

Applying $R_2 \to \frac{R_2}{9}$, we get

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{7}{9} \\ 0 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -2 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + 5R_2$, we get

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & \frac{1}{9} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -\frac{1}{3} & \frac{5}{9} & 1 \end{bmatrix} A$$

Applying $R_3 \to 9 R_3$, we get

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -3 & 5 & 9 \end{bmatrix} A$$

Applying $R_1 \to R_1 - 3R_2$, we get

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & -\frac{7}{9} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{9} & 0 \\ -3 & 5 & 9 \end{bmatrix} A$$

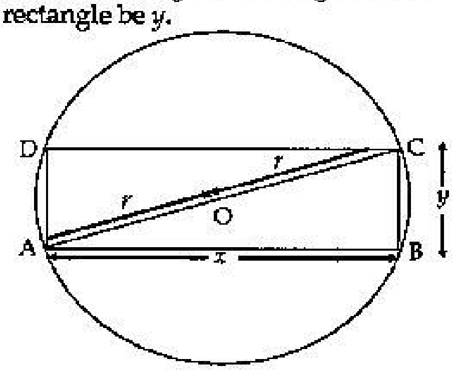
Applying $R_1 \to R_1 - \frac{1}{3} R_3$, $R_2 \to R_2 + \frac{7}{9} R_3$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix} A$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -2 & -3 \\ -2 & 4 & 7 \\ -3 & 5 & 9 \end{bmatrix} \quad \textbf{Ans.}$$

24. Show that of all the rectangles inscribed in a given fixed circle, the square has the maximum area. [6]

Solution : Let length of rectangle be x and breadth of rectangle be y.



Now, area of rectangle

$$A = l \times b = xy \quad ...(i)$$

Let r be radius of circle

In Δ ABC,

$$AC^2 = AB^2 + BC^2$$
$$(2r)^2 = x^2 + y^2$$
$$\Rightarrow \quad 4r^2 - x^2 = y^2 \quad ...(ii)$$
$$A = xy$$
$$\Rightarrow \quad A^2 = x^2y^2 \quad ...(iii)$$

From (ii) and (iii),

$$A^2 = x^2(4r^2 - x^2)$$
$$= 4r^2x^2 - x^4$$

Let $A^2 = f(x)$, $f(x) = 4r^2x^2 - x^4$

Differentiating w.r.t. x, we get

$$f'(x) = 8r^2x - 4x^3 \quad ...(iv)$$

For maximum or minimum,

$$f'(x) = 0$$
$$\Rightarrow \quad 0 = 8r^2x - 4x^3$$
$$\Rightarrow \quad 4x^3 = 8r^2x$$
$$\Rightarrow \quad x^2 = 2r^2$$
$$\Rightarrow \quad x = \sqrt{2}r$$

Again differentiating equation (iv) w.r.t. x, we get

$$f''(x) = 8r^2 - 12x^2$$
$$f''(x)_{x=\sqrt{2}r} = 8r^2 - 12 \times 12r^2$$
$$= 8r^2 - 24r^2$$
$$= -16r^2 < 0.$$

$\therefore$ $f(x)$ or A is maximum at $x = \sqrt{2}\,r$

Putting $r = \frac{x}{\sqrt{2}}$ in equation (ii), we get

$$4\frac{x^2}{2} - x^2 = y^2$$
$$\Rightarrow \quad 2x^2 - x^2 = y^2$$
$$\Rightarrow \quad x^2 = y^2$$
$$\Rightarrow \quad x = y$$

$\therefore$ Rectangle having maximum area is a square.

Hence Proved.

25. Using integration find the area of the triangular region whose sides have equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$. [6]

Solution : Given equations are

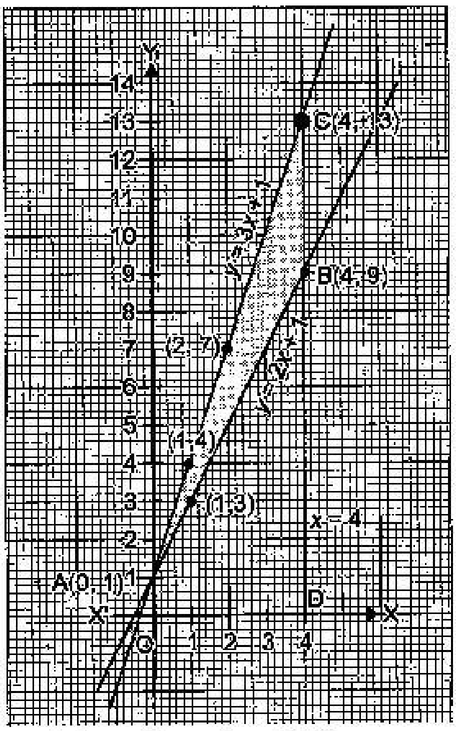
$$y = 2x + 1 \quad ...(i)$$
$$y = 3x + 1 \quad ...(ii)$$

Table for line (i)

x	0	1	2	4
y	1	3	5	9

Table for line (ii),

x	0	1	2	4
y	1	4	7	13



Area of triangular region ABC

= Area of the region OACDO – Area of the region OABDO

$$= \int_0^4 [y \text{ line (ii)} - y \text{ line (i)}]\,dx$$

$$= \int_0^4 [(3x+1)-(2x+1)]\,dx$$

$$= \int_0^4 (3x+1-2x-1) = \int_0^4 x.dx$$

$$= \left[\frac{x^2}{2}\right]_0^4 = \left[\frac{(4)^2}{2}\right] = 8 \text{ sq. units.}$$ **Ans.**

26. Evaluate : $\int_0^{\pi/2} 2\sin x\cos x\tan^{-1}(\sin x)\,dx.$ **[6]**

Solution : Let, $I = \int_0^{\pi/2} 2\sin x\cos x\tan^{-1}(\sin x)\,dx$

Putting $\sin x = t$

$\Rightarrow \quad \cos x\,dx = dt$

If $x = 0, t = 0$

and $x = \frac{\pi}{2}, t = 1$

$$I = 2\int_0^1 t\times\tan^{-1}t.dt$$

$$= 2\left[\frac{t^2}{2}\times\tan^{-1}t\right]_0^1 - 2\int_0^1 \frac{1}{1+t^2}\times\frac{t^2}{2}\,dt$$

$$= 2\times\frac{1}{2}\times\tan^{-1}(1) - \int_0^1 \frac{1+t^2-1}{1+t^2}\,dt$$

$$= 1\times\frac{\pi}{4} - \int_0^1\left(\frac{1+t^2}{1+t^2} - \frac{1}{1+t^2}\right)dt$$

$$= \frac{\pi}{4} - \left[t - \tan^{-1}t\right]_0^1$$

$$= \frac{\pi}{4} - 1 + \tan^{-1}(1) = \frac{2\pi}{4} - 1 = \left(\frac{\pi}{2} - 1\right)$$ **Ans.**

OR

Evaluate : $\int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x + \cos^4 x}\,dx.$

Solution : Let, $I = \int_0^{\pi/2} \frac{x\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$ ...(i)

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}{\sin^4\left(\frac{\pi}{2}-x\right)+\cos^4\left(\frac{\pi}{2}-x\right)}\,dx$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2}-x\right)\cos x.\sin x}{\cos^4 x+\sin^4 x}\,dx.$$ ...(ii)

On adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{x\sin x\cos x + \left(\frac{\pi}{2}-x\right)\cos x.\sin x}{\sin^4 x+\cos^4 x}\,dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\frac{\pi}{2}\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$$

$$= \frac{\pi}{2}.\frac{1}{2}\int_0^{\pi/2} \frac{2\sin x\cos x}{(\sin^2 x)^2+(1-\sin^2 x)^2}\,dx$$

Putting $\sin^2 x = t$

$\Rightarrow \quad 2\sin x\cos x\,dx = dt$

when $x = 0$ then $t = 0$

when $x = \frac{\pi}{2}$ then $t = \sin^2\frac{\pi}{2} = (1)^2 = 1$

$$\therefore \quad 2I = \frac{\pi}{4}\int_0^1 \frac{dt}{t^2+(1-t)^2}$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\int_0^1 \frac{dt}{t^2+1+t^2-2t}$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\int_0^1 \frac{dt}{2t^2-2t+1}$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\int_0^1 \frac{dt}{2\left(t^2-t+\frac{1}{2}\right)}$$

$$\Rightarrow \quad 2I = \frac{\pi}{8}\int_0^1 \frac{dt}{\left(t-\frac{1}{2}\right)^2-\frac{1}{4}+\frac{1}{2}}$$

$$\Rightarrow \quad 2I = \frac{\pi}{8}\int_0^1 \frac{dt}{\left(t-\frac{1}{2}\right)^2+\frac{1}{4}}$$

$$\Rightarrow \quad 2I = \frac{\pi}{8}\int_0^1 \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$$

$$\Rightarrow \quad 2I = \frac{\pi}{8}\cdot\frac{1}{1/2}\left[\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\left[\tan^{-1}\left(\frac{1/2}{1/2}\right)-\tan^{-1}\left(\frac{-1/2}{1/2}\right)\right]$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}[\tan^{-1}(1)+\tan^{-1}(1)]$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\left[\frac{\pi}{4}+\frac{\pi}{4}\right]$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}\left[\frac{\pi}{2}\right]$$

$$\Rightarrow \quad 2I = \frac{\pi^2}{8}$$

$$\therefore \quad I = \frac{\pi^2}{16}.$$ **Ans.**

27. Find the equation of the plane which contains the line of intersection of the planes $\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4=0$, $\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8=0$. [6]

Solution : The given equations are

$$\vec{r}.(\hat{i}+2\hat{j}+3\hat{k})-4 = 0$$

$$\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5 = 0$$

The equation of plane containing them is

$$\vec{r}.[\hat{i}+2\hat{j}+3\hat{k}]-4+\lambda(\vec{r}.(2\hat{i}+\hat{j}-\hat{k})+5)=0 \quad ...(i)$$

$$\Rightarrow \quad \vec{r}.[(1+2\lambda)\hat{i}+(2+\lambda)\hat{j}+(3-\lambda)\hat{k}]=4-5\lambda$$

This is perpendicular to plane

$$\vec{r}.(5\hat{i}+3\hat{j}-6\hat{k})+8=0$$

$$\Rightarrow \quad (1+2\lambda)\times 5+(2+\lambda)\times 3+(3-\lambda)(-6) = 0$$

$$\Rightarrow \quad 5+10\lambda+6+3\lambda-18+6\lambda = 0$$

$$\Rightarrow \quad \lambda = \frac{7}{19}$$

Put $\lambda=\frac{7}{19}$ in equation (i), we get

The required equation of the plane is

$$\vec{r}.\left[\frac{33}{19}\hat{i}+\frac{45}{19}\hat{j}+\frac{50}{19}\hat{k}\right]=\frac{41}{19}$$

$$\Rightarrow \quad \vec{r}.(33\hat{i}+45\hat{j}+50\hat{k}) = 41.$$ **Ans.**

28. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftsman's time in its making while a cricket bat takes 3 hours of machine time and 1 hour of craftsman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time. If the profit on a racket and on a bat is ₹ 20 and ₹ 10 respectively, find the number of tennis rackets and circket bats that the factory must manufacture to earn the maximum profit. Make it as an L.P.P. and solve graphically. [6]

Solution : Let x be the number of tennis rackets and y that of cricket bats produced in one day in the factory.

Item	Number	Machine hours	Craftsman hours	Maximize (Profit)
Tennis Racket	x	1.5	3	₹ 20
Cricket Bats	y	3	1	₹10
Total		42	24	

Maximize $Z = 20x + 10y$

Subject to constraints :

$$1.5x+3y \le 42$$

$$3x+y \le 24$$

$$x \ge 0, y \ge 0$$

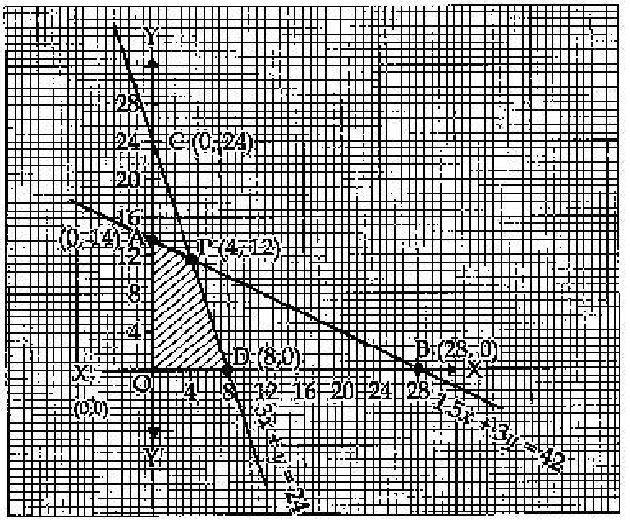
First we draw the lines AB and CD whose equations are

$$1.5x+3y = 42 \quad ...(i)$$

	A	B
x	0	28
y	14	0

and $\quad 3x + y = 24 \quad$ ...(ii)

	C	D
x	0	8
y	24	0



The feasible region is ODPAO which is shaded in the figure.

The vertices of the feasible region are O(0, 0), D(8, 0), P (4, 12) and A(0, 14).

P is the point of intersection of the lines

$$1.5x + 3y = 42 \text{ and } 3x + y = 24$$

Solving these equations, we get point P(4, 12).

The value of objective function $Z = 20x + 10y$ at these vertices are as follows :

Corner Points	$Z = 20x + 10y$
O (0, 0)	0
D (8, 0)	$20 \times 8 + 0 = 160$
P (4, 12)	$20 \times 4 + 10 \times 12$ $= 200$ (maximum)
A (0, 14)	$0 + 10 \times 14 = 140$

For maximum profit ₹ 200, 4 tennis rackets and 12 cricket bats should be produced. **Ans.**

29. Suppose 5% of men and 0.25% of women have grey hair. A grey haired person is selected at random. What is the probability of this person being male ? Assume that there are equal number of males and females. [6]

Solution : Let E_1 and E_2 be the number of men and women respectively.

$\therefore$ Probability of men and women

$$P(E_1) = P(E_2) = \frac{1}{2} \text{ or } 0.5$$

Let A be event of selecting a grey person.

$$P(A/E_1) = 5\% = 0.05$$

$$P(A/E_2) = 0.25\% = 0.0025$$

$\therefore$ Probability of person being male (By Bayes' Theorem)

$$P(E_1/A) = \frac{P(E_1) \times P(A/E_1)}{P(E_1) \times P(A/E_1) + P(E_2) \times P(A/E_2)}$$

$$= \frac{0.5 \times 0.05}{0.5 \times 0.05 + 0.5 \times 0.0025}$$

$$= \frac{0.025}{0.025 + 0.00125}$$

$$= \frac{0.025}{0.02625} = 0.95.$$ **Ans.**

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Mathematics 2011 (Delhi) SET II

Time allowed : 3 hours **Maximum marks : 100**

> **Note : Except for the following questions, all the remaining questions have been asked in previous set.**

SECTION — A

9. Write the value of $\tan^{-1}\left[\tan\frac{3\pi}{4}\right]$. [1]

Solution : We know that $\tan^{-1}(\tan x) = x$ if $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal value branch of $\tan^{-1} x$.

Here, $\frac{3\pi}{4} \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Now, $\tan^{-1}\left[\tan\frac{3\pi}{4}\right]$ can be written as

$$\tan^{-1}\left[\tan\frac{3\pi}{4}\right] = \tan^{-1}\left[\tan\left(\pi - \frac{\pi}{4}\right)\right]$$

$$= \tan^{-1}\left[-\tan\frac{\pi}{4}\right]$$

$$= \tan^{-1}\left[\tan\left(-\frac{\pi}{4}\right)\right] = \frac{-\pi}{4}$$

where $\frac{-\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\therefore \tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}\left[\tan\left(\frac{-\pi}{4}\right)\right] = \frac{-\pi}{4}.$$

Ans.

10. Write the value of $\int \frac{\sec^2 x}{\text{cosec}^2 x} dx.$ **[1]**

Solution : Given $\int \frac{\sec^2 x}{\text{cosec}^2 x} dx = \int \frac{\sin^2 x}{\cos^2 x} dx$

$$= \int \tan^2 x\, dx = \int (\sec^2 x - 1) dx$$

$$(\because \sec^2 x - \tan^2 x = 1)$$

$$= \int \sec^2 x\, dx - \int 1.dx = \tan x - x + C.$$ **Ans.**

SECTION — B

15. Form the differential equation of the family of parabolas having vertex at the origin and axis along positive y-axis. **[4]**

Solution : The equation of the family of parabolas having vertex at the origin and axis along positive y-axis.

$$x^2 = 4ay \quad ...(i)$$

Differentiating w.r.t. x, we get

$$2x = 4a\frac{dy}{dx}$$

$$\Rightarrow \quad 2x.\frac{dx}{dy} = 4a \quad ...(ii)$$

Putting the value of $4a$ in equation (i), we get

$$x^2 = \left(2x.\frac{dx}{dy}\right)(y) \Rightarrow x\frac{dy}{dx} = 2y.$$ **Ans.**

16. Find a vector of magnitude 5 units and parallel to the resultant of the vectors $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-2\hat{j}+\hat{k}$. **[4]**

Solution : We have $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{b} = \hat{i}-2\hat{j}+\hat{k}$.

Let $\vec{c}$ be the resultant of $\vec{a}$ and $\vec{b}$, then

$$\vec{c} = \vec{a}+\vec{b}$$

$$= (2\hat{i}+3\hat{j}-\hat{k})+(\hat{i}-2\hat{j}+\hat{k})$$

$$= 3\hat{i}+\hat{j}+0\hat{k}$$

$$\therefore \quad |\vec{c}| = \sqrt{3^2+1^2+0} = \sqrt{9+1} = \sqrt{10}$$

Now, unit vector in the direction of $\vec{c}$ is

$$\therefore \quad \hat{c} = \frac{\vec{c}}{|\vec{c}|} = \frac{3\hat{i}+\hat{j}}{\sqrt{10}}$$

Hence, the vector of magnitude 5 units and parallel to the resultant of vector $\vec{a}$ and $\vec{b}$ is

$$\pm 5\hat{c} = \pm 5.\frac{(3\hat{i}+\hat{j})}{\sqrt{10}}.$$ **Ans.**

19. If the function $f(x)$ given by

$$f(x) = \begin{cases} 3ax+b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax-2b, & \text{if } x < 1 \end{cases}$$ **is continuous at**

$x = 1$, find the values of a and b. **[4]**

Solution : Since f is continuous at $x = 1$, therefore,

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x) = f(1) \quad ...(i)$$

$$\text{L.H.L.} = \lim_{x\to1^-} (5ax-2b) = 5a-2b$$

$$\text{R.H.L.} = \lim_{x\to1^+} (3ax+b) = (3a+b)$$

$$\therefore \quad 5a-2b = 3a+b = 11 \quad \text{[Using (i)]}$$

$$3a+b = 11 \quad ...(ii)$$

and $$5a-2b = 11 \quad ...(iii)$$

Multiplying (ii) by 2 and adding it to (iii), we get

$$2(3a+b)+5a-2b = (2\times11)+11$$

$$\Rightarrow \quad 6a+2b+5a-2b = 22+11$$

$$\Rightarrow \quad 11a = 33$$

$$\Rightarrow \quad a = 3$$

Substituting $a = 3$ in (ii), we get

$$3\times3+b = 11$$

$$\Rightarrow \quad b = 11-9 = 2$$

Thus, $a = 3$ and $b = 2$ will make $f(x)$ continuous at $x = 1$. **Ans.**

20. Using properties of determinants, prove the following :

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x).$$ **[4]**

Solution : L.H.S. $= \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$

Taking common x, y, z from C_1, C_2 and C_3, we get

$$= xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$; $C_2 \to C_2 - C_3$, we get

$$\Delta = xyz \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$$

Expanding along C_3, we get

$$= xyz \begin{vmatrix} x-y & y-z \\ x^2-y^2 & y^2-z^2 \end{vmatrix}$$

Taking $(x-y)$ common from C_1 and $y-z$ common from C_2

$$= xyz(x-y)(y-z) \begin{vmatrix} 1 & 1 \\ x+y & y+z \end{vmatrix}$$

$$= xyz\,(x-y)\,(y-z)\,(y+z-x-y)$$

$$= xyz\,(x-y)\,(y-z)\,(z-x) = \text{R.H.S.}$$

Hence Proved.

SECTION — C

23. Bag I contains 3 red and 4 black balls and Bag II contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from Bag II. **[6]**

Solution :

Bag I	
R	B
3	4

Bag II	
R	B
5	6

Let E_1 be the event that Bag I is chosen and E_2 be the event that Bag II is chosen.

Let A be the event that the chosen ball is red

$\therefore \quad P(E_1) = \frac{1}{2} = P(E_2)$

$\therefore$ Probability of red ball from Bag I,

$$P(A/E_1) = \frac{3}{3+4} = \frac{3}{7};$$

and probability of red ball from Bag II,

$$P(A/E_2) = \frac{5}{5+6} = \frac{5}{11}$$

By Bayes' theorem,

$$P(E_2/A) = \frac{P(E_2).P(A/E_2)}{P(E_1).P(A/E_1)+P(E_2).P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{5}{11}}{\frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{5}{11}} = \frac{\frac{5}{11}}{\frac{3}{7}+\frac{5}{11}}$$

$$= \frac{\frac{5}{11}}{\frac{33+35}{77}} = \frac{\frac{5}{11}}{\frac{68}{77}} = \frac{5\times77}{68\times11} = \frac{35}{68}. \qquad \textbf{Ans.}$$

29. Show that of all the rectangles with a given perimeter, the square has the largest area. **[6]**

Solution : Let x and y be the length and breadth of the rectangle whose perimeter is given $4a$ (say)

$\therefore \quad$ Area, $A = xy$ ...(i)

$2x + 2y = 4a$ (Given)

$\Rightarrow \quad y = 2a - x$ ...(ii)

Putting y in (i), we get

$A = x(2a - x)$

$\therefore \quad A = 2ax - x^2$ ...(iii)

Differentiating w.r.t. x, we get

$\therefore \quad \frac{dA}{dx} = 2a - 2x$ ...(iv)

For maxima or minima,

$\frac{dA}{dx} = 0$

$\therefore \quad 2a - 2x = 0$

$\Rightarrow \quad 2x = 2a$

$\Rightarrow \quad x = a$

Again differentiating w.r.t. x, we get

$\frac{d^2A}{dx^2} = -2$

$\therefore \quad \left[\frac{d^2A}{dx^2}\right]_{\text{at } x=a} = -2 < 0$

$\therefore$ Area A is maximum at $x = a$

$\therefore \quad y = 2a - a$

$\Rightarrow \quad y = a$

Hence, It is proved that all the rectangles with a given perimeter, the square has the largest area.

Hence Proved.

Mathematics 2011 (Delhi) SET III

Time allowed : 3 hours **Maximum marks : 100**

> **Note : Except for the following questions, all the remaining questions have been asked in previous sets.**

SECTION — A

1. Write the value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$. **[1]**

Solution : We know that $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$, which is the principal value branch of $\cos^{-1} x$.

Here, $\frac{7\pi}{6} \notin [0, \pi]$.

Now, $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ can be written as :

$$\cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left[\cos\left(2\pi - \frac{7\pi}{6}\right)\right]$$

$[\because \cos(2\pi - x) = \cos x]$

$$= \cos^{-1}\left[\cos\frac{5\pi}{6}\right],$$

where $\frac{5\pi}{6} \in [0, \pi]$

$$\therefore \cos^{-1}\left(\cos\frac{7\pi}{6}\right) = \cos^{-1}\left(\cos\frac{5\pi}{6}\right) = \frac{5\pi}{6}.$$ **Ans.**

2. Write the value of $\int \frac{2 - 3\sin x}{\cos^2 x} dx$. **[1]**

Solution : Let $I = \int \frac{2 - 3\sin x}{\cos^2 x} dx$.

$$= \int\left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos^2 x}\right)dx$$

$$= \int 2\sec^2 x\, dx - 3\int \tan x \sec x\, dx$$

$$= 2\tan x - 3\sec x + C.$$ **Ans.**

SECTION — B

11. Using properties of determinants, prove the following :

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2.$$ **[4]**

Solution : Taking L.H.S.

$$\text{Let } \Delta = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we get

$$\Delta = (5x+4)\begin{vmatrix} 1 & 0 & 0 \\ 2x & -x+4 & 0 \\ 2x & 0 & -x+4 \end{vmatrix}$$

Taking $(4 - x)$ common from C_2 and C_3, we get

$$= (5x+4)(4-x)(4-x)\begin{vmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 2x & 0 & 1 \end{vmatrix}$$

Expanding along C_3, we get

$$\Delta = (5x+4)(4-x)^2\begin{vmatrix} 1 & 0 \\ 2x & 1 \end{vmatrix}$$

$$= (5x+4)(4-x)^2 = \text{R.H.S.}$$

Hence Proved.

12. Find the value of a and b such that the following function $f(x)$ is a continuous function :

$$f(x) = \begin{cases} 5, & x \le 2 \\ ax + b, & 2 < x < 10 \\ 21, & x \ge 10 \end{cases}$$ **[4]**

Solution : If f is a continuous function, f is continuous at all real numbers.

In particular, $f(x)$ is continuous at $x = 2$ and $x = 10$.

Since f is continuous at $x = 2$, we obtain

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$$

$$\Rightarrow \lim_{x\to 2^-} (5) = \lim_{x\to 2^+} (ax+b) = 5$$

$$\Rightarrow 5 = 2a + b = 5$$

$$\Rightarrow 2a + b = 5 \quad \text{...(i)}$$

Since f is continuous at $x = 10$, we obtain

$$\lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$$

$\Rightarrow \quad \lim_{x\to 10^-}(ax+b) = \lim_{x\to 10^+}(21) = 21$

$\Rightarrow \quad 10a + b = 21 \quad ...(ii)$

On subtracting equation (i) from equation (ii), we obtain

$8a = 16$

$\Rightarrow \quad a = 2$

By putting $a = 2$ in equation (i), we obtain

$2 \times 2 + b = 5$

$\Rightarrow \quad 4 + b = 5$

$\Rightarrow \quad b = 1$

Therefore, the values of a and b for which $f(x)$ is a continuous function are 2 and 1 respectively.

Ans.

13. Solve the following differential equation :

$(1 + y^2)(1 + \log x)\, dx + x dy = 0$ [4]

Solution : Given $(1 + y^2)(1 + \log x)\, dx + x dy = 0$

$\Rightarrow (1 + y^2)(1 + \log x)\, dx = -x dy$

$\Rightarrow \quad \frac{1}{x}(1+\log x)dx = \frac{-1}{(1+y^2)}dy$

Integrating on both sides, we get

$$\int \frac{1}{x}(1+\log x)dx = \int -\frac{1}{(1+y^2)}dy \quad ...(i)$$

Let $\quad \log x = t$

$\Rightarrow \quad \frac{1}{x}dx = dt$

Now, from (i)

$$\int (1+t)dt = -\int \frac{1}{1+y^2}dy$$

$$\Rightarrow \quad t + \frac{t^2}{2} = -\tan^{-1} y + C_1$$

$$\Rightarrow \quad \log x + \frac{(\log x)^2}{2} = -\tan^{-1} y + C_1$$

$\Rightarrow \quad 2\log x + (\log x)^2 = -2\tan^{-1} y + 2C_1$

$\Rightarrow (\log x)^2 + 2\log x + 2\tan^{-1} y - 2C_1 = 0$

$\Rightarrow (\log x)^2 + 2\log x + 2\tan^{-1} y + C = 0$, where $-2C_1 = C$

Ans.

14. If two vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a}.\vec{b} = 1$, then find the value of $(3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b})$. [4]

Solution : It is given that $|\vec{a}| = 2$, $|\vec{b}| = 1$ and $\vec{a}.\vec{b} = 1$

$\therefore \; (3\vec{a} - 5\vec{b}).(2\vec{a} + 7\vec{b})$

$= 3\vec{a}.2\vec{a} + 3\vec{a}.7\vec{b} - 5\vec{b}.2\vec{a} - 5\vec{b}.7\vec{b}$

$= 6|\vec{a}|^2 + 21(\vec{a}.\vec{b}) - 10(\vec{b}.\vec{a}) - 35|\vec{b}|^2$

$= 6(2)^2 + 21(1) - 10(1) - 35(1)^2 \quad [\vec{b}.\vec{a} = \vec{a}.\vec{b}]$

$= 24 + 21 - 10 - 35 = 0.$ **Ans.**

SECTION – C

23. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is a six. Find the probability that it is actually a six. [6]

Solution : Let E be the event that the man reports that six occurs in the throwing of a die and let S_1 be the event that six occurs and S_2 be the event that six does not occur,

$P(S_1)$ = Probability that six occurs = $\frac{1}{6}$

$P(S_2)$ = Probability that six does not occur = $\frac{5}{6}$

$P(E/S_1)$ = Probability that the man reports that six occurs when six has actually occurred on the die
= Probability that the man speaks the truth
= $\frac{3}{4}$

$P(E/S_2)$ = Probability that the man reports that six occurs when six has not actually occurred on the die
= Probability that the man does not speaks the truth
= $1 - \frac{3}{4} = \frac{1}{4}$

Using Bayes' theorem,

$P(S_1/E)$ = Probability that the report of the man that six has occurred is actually a six

$$= \frac{P(S_1)P(E/S_1)}{P(S_1)P(E/S_1) + P(S_2)P(E/S_2)}$$

$$= \frac{\frac{1}{6}\times\frac{3}{4}}{\frac{1}{6}\times\frac{3}{4} + \frac{5}{6}\times\frac{1}{4}} = \frac{1}{8}\times\frac{24}{8} = \frac{3}{8}$$

Thus, the required probability is $\frac{3}{8}$. **Ans.**

24. Show that of all the rectangles of given area, the square has the smallest perimeter. [6]

Solution : Let l and b respectively be the length and the breadth of the rectangle of given area A.

$A = l \times b$

$\Rightarrow \quad b = \frac{A}{l} \quad ...(i)$

Perimeter of the rectangle, $P = 2(l + b)$

$\therefore \quad P = 2\left(l + \frac{A}{l}\right)$

Differentiating w.r.t. l, we get

$$\frac{dP}{dl} = 2\left(1-\frac{A}{l^2}\right) \qquad \text{...(ii)}$$

Again differentiating, we get

$$\frac{d^2P}{dl^2} = 2\left(0+\frac{2A}{l^3}\right)$$

$$\Rightarrow \quad \frac{d^2P}{dl^2} = \frac{4A}{l^3} \qquad \text{...(iii)}$$

For maximum or minimum perimeter, $\frac{dP}{dl} = 0$

$$\Rightarrow \quad 2\left(1-\frac{A}{l^2}\right) = 0$$

$$\Rightarrow \quad 1-\frac{A}{l^2} = 0$$

$$\Rightarrow \quad \frac{A}{l^2} = 1$$

$$\Rightarrow \quad A = l^2$$

Substituting the value of A in equation (i), we get

$$b = \frac{A}{l} = \frac{l^2}{l} = l$$

$$\therefore \quad b = l = \sqrt{A}$$

From (iii),

$$\frac{d^2P}{dl^2} = \frac{4l^2}{l^3}$$

$$\frac{d^2P}{dl^2} = \frac{4}{l} = \frac{4}{\sqrt{A}}$$

The value of $\sqrt{A}$ cannot be negative,

$$\therefore \quad \frac{d^2P}{dl^2} > 0$$

Hence, of all the rectangles of given area, the square has the smallest perimeter.

Hence Proved.